Two "Negative" Theorems

CPSCI 330: Algorithms
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1 Exponential Functions

Theorem N1
Exponential functions with different positive bases have different orders of growth.

Prove: $\boldsymbol{\Theta}(a^n) \neq \boldsymbol{\Theta}(b^n)$ for $a, b > 0, a \neq b$.

1.1 Proof Strategy

By Theorem 3, we know that $\boldsymbol{\Theta}(g(n)) = \boldsymbol{O}(g(n)) \cap \boldsymbol{\Omega}(g(n))$. So if we can show that some function in $\boldsymbol{\Theta}(b^n)$ is not a member of $\boldsymbol{O}(a^n)$ or $\boldsymbol{\Omega}(a^n)$, then that function can't be in $\boldsymbol{\Theta}(a^n)$, and hence $\boldsymbol{\Theta}(a^n) \neq \boldsymbol{\Theta}(b^n)$.

Without loss of generality it may be assumed that $b > a$. So we can prove Theorem N1 if we can show the following:

Prove: $b^n \notin \boldsymbol{O}(a^n)$ for $b > a > 0$.

1.2 Proof

Suppose that $b^n \in \boldsymbol{O}(a^n)$. Then by definition of $\boldsymbol{O}$, there exists a positive constant c and a nonnegative integer n_0 such that

$$b^n \leq c \cdot a^n \quad {}_{\forall n \geq n_0}$$

Isolate c:

$$\frac{b^n}{a^n} \leq c \quad {\scriptstyle \forall n \geq n_0}$$

$$\left(\frac{b}{a}\right)^n \leq c \quad {\scriptstyle \forall n \geq n_0} \tag{1}$$

Since $b > a$, it must be the case that $\frac{b}{a} > 1$. Hence, $\forall n \geq 0$, $\left(\frac{b}{a}\right)^{n+1} > \left(\frac{b}{a}\right)^n$. In other words, $\left(\frac{b}{a}\right)^n$ is uniformly increasing. Hence, for any positive constant c, $\exists n \geq n_0$ such that $\left(\frac{b}{a}\right)^n > c$. But this contradicts (1), above. Hence, $b^n \notin \boldsymbol{O}(a^n)$.

2 Which Side Are You On?

Given: Two arbitrary nonnegative functions defined on the nonnegative integers, $t(n)$ and $g(n)$.

Hypothesis: Either $t(n) \in \boldsymbol{O}(g(n))$, or $t(n) \in \boldsymbol{\Omega}(g(n))$, or both.

2.1 Proof Strategy

If you think the hypothesis is true, you must show that for **any** $t(n)$ and $g(n)$ on the nonnegative integers, $t(n) \in \boldsymbol{O}(g(n))$ **or** $t(n) \in \boldsymbol{\Omega}(g(n))$. (You don't have to show both—that would be a different, stronger conclusion.)

If you think the hypothesis is false, you must show that for **some** $t(n)$ and $g(n)$ on the nonnegative integers, **both** $t(n) \notin \boldsymbol{O}(g(n))$ **and** $t(n) \notin \boldsymbol{\Omega}(g(n))$.

Take a few moments to think carefully about this hypothesis. It may help to sketch the graphs of a few nonnegative functions on the nonnegative integers as you do your thinking. Don't turn to the next page until you've made a confident decision on the truth of the hypothesis.

It turns out that the hypothesis is false. To prove this, we must find two specific counterexample functions *and* show that these functions do, in fact, provide a counterexample.

2.2 Counterexample

Here are two counterexample functions.

$$t(n) = \begin{cases} n & n \text{ is even} \\ n^2 & \text{otherwise} \end{cases}$$

$$g(n) = \begin{cases} n^2 & n \text{ is even} \\ n & \text{otherwise} \end{cases}$$

Before examining the proof, graph the two functions to get a feeling for what they look like and why they provide a counterexample to the hypothesis.

It would also be a good idea to try to work out the proof on your own before reading further.

2.3 Proof

To show:

1. $t(n) \notin \boldsymbol{O}(g(n))$
2. $t(n) \notin \boldsymbol{\Omega}(g(n))$

2.3.1 Proof of 1 by Contradiction

Suppose that $t(n) \in \boldsymbol{O}(g(n))$. Then $\exists$ a positive constant c, nonnegative integer n_0 such that $\forall n \geq n_0$,

$$t(n) \leq c \cdot g(n)$$

Pick an **odd** w such that $w > n_0$ and $w > c$. It follows that

$$\begin{aligned} t(w) &\leq c \cdot g(w) \\ w^2 &\leq c \cdot w \\ w &\leq c \end{aligned}$$

But this is impossible, since we specifically picked $w > c$. So c does not exist, and hence $t(n) \notin \boldsymbol{O}(g(n))$.

2.3.2 Proof of 2 by Contradiction

Suppose that $t(n) \in \boldsymbol{\Omega}(g(n))$. Then $\exists$ a positive constant c, nonnegative integer n_0 such that $\forall n \geq n_0$,

$$c \cdot g(n) \leq t(n)$$

Pick an **even** m such that $m > n_0$. It follows that

$$\begin{aligned} c \cdot g(m) &\leq t(m) \\ c \cdot m^2 &\leq m \\ c &\leq \frac{1}{m} \end{aligned} \tag{2}$$

Note that this is the case for *any* even $m > n_0$.

Since c is a positive constant, it must be the case that there exists a positive integer $r \geq n_0$ such that $0 < \frac{1}{r} < c$. Further, if $\frac{1}{r} < c$, then $\frac{1}{2r} < c$.

Note, however, that $2r$ is both even and exceeds n_0. Hence by (2), $c \leq \frac{1}{2r}$. But this is impossible because of how we chose r. So c does not exist, and hence $t(n) \notin \boldsymbol{\Omega}(g(n))$.

2.3.3 Conclusion

We have specified two nonnegative functions $t(n)$ and $g(n)$ on the nonnegative integers, and shown both that $t(n) \notin \boldsymbol{O}(g(n))$ and $t(n) \notin \boldsymbol{\Omega}(g(n))$. This disproves the Hypothesis.