Theorems for Efficiency Analysis

CPSCI 330: Algorithms
Brian Rosmaita

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1 Some Important Theorems

Theorem 1 (Levitin, p. 56)
If $t_1(n) \in \boldsymbol{O}(g_1(n))$ *and* $t_2(n) \in \boldsymbol{O}(g_2(n))$*, then* $t_1(n)+t_2(n) \in \boldsymbol{O}(max\{g_1(n), g_2(n)\})$.

Corollary 1.1 Theorem 1 holds for $\boldsymbol{\Omega}$ and $\boldsymbol{\Theta}$.

Theorem 2
$\boldsymbol{\Theta}(\alpha g(n)) = \boldsymbol{\Theta}(g(n))$ *where* $\alpha > 0$.

Corollary 2.1 $\boldsymbol{\Theta}(\log_b n) = \boldsymbol{\Theta}(\log_a n)$ for $a, b > 1$.

Theorem 3
$\boldsymbol{\Theta}(g(n)) = \boldsymbol{O}(g(n)) \cap \boldsymbol{\Omega}(g(n))$.

Corollary 3.1 If $t(n) \in \boldsymbol{O}(g(n))$ and $t(n) \in \boldsymbol{\Omega}(g(n))$, then $t(n) \in \boldsymbol{\Theta}(g(n))$.

Theorem 4
Let $p(n)$ *be a nonnegative polynomial of degree* k*. Then* $p(n) \in \boldsymbol{\Theta}(n^k)$.

2 Proof of Corollary 2.1

Logarithms have an interesting property listed on page 469 in your text, which, rewritten for our purposes, states:

$$\log_b n = \log_b a \cdot \log_a n$$

Note that when we restrict $a, b > 1$ (as in the statement of the Corollary), it follows that $\log_b a > 0$ (i.e., $\log_b a$ is a positive constant). The Corollary then follows by Theorem 2.

3 Proof of Theorem 4

3.1 Proof Strategy

From Corollary 3.1, it follows that the Theorem will be proved if we can establish:

1. $p(n) \in \boldsymbol{O}(n^k)$
2. $p(n) \in \boldsymbol{\Omega}(n^k)$

3.2 Proof of Part 1

We begin with the fact that $p(n)$ is a polynomial of degree k.

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

If we restrict n such that $n \geq 0$, it follows that

$$\leq |a_k| n^k + |a_{k-1}| n^{k-1} + \cdots + |a_1| n + |a_0|$$

If we further restrict n such that $n \geq 1$,

$$\leq |a_k| n^k + |a_{k-1}| n^k + \cdots + |a_1| n^k + |a_0| n^k$$

Finally, by distribution,

$$\leq (|a_k| + |a_{k-1}| + \cdots + |a_1| + |a_0|) n^k$$

Let $c = |a_k| + |a_{k-1}| + \cdots + |a_1| + |a_0|$, and let $n_0 = 1$. Thus we have a positive constant c and a nonnegative integer n_0 such that $p(n) \leq cn^k$ for all $n \geq n_0$. Hence, $p(n) \in \boldsymbol{O}(n^k)$, which proves Part 1.

3.3 Proof of Part 2

3.3.1 Proof Strategy

Consider again our nonnegative polynomial of degree k:

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

If we factor out n^k from each term, we get

$$= n^k \big(a_k + \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}\big) \tag{1}$$

We need to find a positive constant c and a nonnegative integer n_0 such that $p(n) \geq cn^k$ for all $n \geq n_0$. In expression 1, above, the sum consisting of all terms with n^i in the denominator goes to zero for large n, so if we set n_0 large enough, we can ignore everything except a_k as a multiplier for n^k. So we need an expression for c that's smaller than a_k. How about $\frac{a_k}{2}$? We make the following observations:

1. If a_k is positive, then $a_k > \frac{a_k}{2}$.
2. If a_k is positive and $n_0 \geq 0$, then $a_k n^k \geq \frac{a_k}{2} n^k$.
3. $\lim\limits_{n \to \infty} \left(\frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \right) = 0$

So if we pick n_0 large enough, and can show that $a_k > 0$, it looks like $\frac{a_k}{2}$ will be a good candidate for the constant c in the definition of $\mathbf{\Omega}$.

3.3.2 Proof that $a_k > 0$

Let's assume that $a_k < 0$ and show that this entails a contradiction.

If $a_k < 0$, then there exists an $\epsilon > 0$ such that $a_k + \epsilon < 0$. By observation 3, above, we know that for sufficiently large n,

$$\left| \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \right| < \epsilon.$$

Repeating expression (1), above, we know that

$$p(n) = n^k \big(a_k + \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \big)$$

Taking the absolute value of the terms with n^i in the denominator will either increase their sum or leave it the same, so

$$p(n) \leq n^k \big(a_k + \big| \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \big| \big)$$

Further,

$$p(n) < n^k(a_k + \epsilon)$$

since by specification, ϵ is strictly greater than the quantity it replaces. Recall that we picked ϵ such that $a_k + \epsilon < 0$, and we're working with large n, i.e., $n \gg 0$. Hence, n^k is positive, but $n^k(a_k + \epsilon)$ is negative, so

$$p(n) < n^k(a_k + \epsilon) < 0$$

i.e.,

$$p(n) < 0$$

But this contradicts the fact that $p(n)$ is a nonnegative polynomial. So we reject the assumption and conclude that $a_k \not< 0$. Further, because $p(n)$ is of degree k, we know that $a_k \neq 0$. Hence, $a_k > 0$.

3.3.3 Determining n_0

Finally, we need to show that $\frac{a_k}{2}$ will work as the constant c in the definition of $\mathbf{\Omega}$. We do this by finding a suitable n_0.

We've just shown that $a_k > 0$. Recalling the "limit" observation (observation 3, above), it follows that there exists an n_0 such that for all $n \geq n_0$,

$$\frac{a_k}{2} > \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \tag{2}$$

This is the n_0 we want. We now show why it will work. Recall expression (1), namely,

$$p(n) = n^k \big(a_k + \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} \big)$$

For all $n \geq n_0$,

$$p(n) \geq n^k (a_k - \frac{a_k}{2})$$

since, by expression (2), we are now subtracting a larger quantity. Simplifying, we obtain,

$$p(n) \geq n^k \big(\frac{a_k}{2} \big) \tag{3}$$

Let $\frac{a_k}{2} = c$, n_0 as above. Making appropriate substitution in expression (3), it follows by the definition of $\mathbf{\Omega}$,

$$p(n) \in \mathbf{\Omega}(n^k).$$

3.4 Conclusion

It follows by Corollary 3.1 that for any nonnegative polynomial in n of degree k, $p(n) \in \mathbf{\Theta}(n^k)$.