Econ 460
Game Theory
Assignment 4
Games in Extensive Form, Backward Induction, Subgame Perfection (Ch. 14,15), Bargaining (Ch. 19), Finitely Repeated Games (Ch. 22)

Games in Extensive Form, Backward Induction, and Subgame Perfection:

1. **Ultimatum Game:** A social scientist (let's call him Bob) walks up to you on a sidewalk and pulls out a wad of 100 one dollar bills. He stops a second pedestrian and proposes the following game. You are to propose a division of the wad of bills (between you and the second pedestrian) and then the second pedestrian either accepts the division or does not accept. If she accepts, you split the money according to your proposal. If she does not accept, Bob keeps the money.

   a) What does the game tree for this game look like?
   b) Find a backward induction solution to the game.
   c) How well does this solution describe outcomes of ultimatum game experiments?

2. **Centipede Game:** A social scientist (let's call her Beth) walks up to you on a sidewalk and pulls out a single dollar bill. She stops a second pedestrian and proposes the following game. You get to decide whether to take the dollar, at which point the game ends, or pass this round, in which case Beth adds a second dollar and turns to the second pedestrian, who can either take the two dollars or pass, and so on. Each time a player passes, Beth will add a dollar to the pot and play moves back to the other player. If the pot reaches $10, that round will be the last. What does the game tree for this game look like? What is the backward induction solution to this game? Is that outcome efficient?

3. **A Nim-like Game:** Two players (Lisa and Bart) take turns choosing a number between 1 and 10 (inclusive). Lisa goes first. The player who takes the total to exactly 100 is the winner. Who should win this game?

4. **Stackelberg Leadership:** Consider again the Cournot Duopoly problem from Assignment 2 #2 part a. Suppose now that we recast this as a sequential move game in which firm I sets its level of output q_1 before firm II sets its level q_2. Assume that II sees I's action before it acts (i.e., this is a game of perfect information). In the subgame perfect Nash equilibrium of this game, what are the quantities set by each firm and how much profit is made by each firm? What is the market quantity and price at this equilibrium? Does this outcome correspond to one of the cases in the Cournot problem?

5. **A Self Control Problem:** Odysseus wanted to listen to the Sirens' song, but knew that he would be overcome by it and would drive his ship into the rocks. Here we can think of this as a game between Odysseus' current self and his future self, where the current self knows that the future self has different preferences than the current self. This is one way of thinking about addictive behavior and problems of "self control" in general. Today we might prefer to put off studying for an exam until tomorrow, but know that tomorrow we will want to put it off again. We might want to have a few drinks or do a little gambling,

but know that once we get started, we will not want to quit at moderation. In these cases we know that we have a tendency to make certain decision that we will regret later.

Odysseus could have avoided the island of the Sirens altogether. Instead he instructed his crew to lash him to the mast and plug their (the crew members') ears with wax so that he could sail past and hear the song without self destructing.

Here is a simple representation of this problem as a game. This morning, Odysseus can choose to sail toward the island of the Sirens (T) or away from it (A). If he sails away, then there is no further choice to be made and his payoffs are (0,0), where the first number is how he values the outcome now, and the second number is how he will value the outcome this afternoon. If he chooses T this morning, he will be near the island this afternoon and will hear the siren song. At that point, he will choose whether to sail past (P) the island safely or sail toward (T) the island, dashing the ship into the rocks. The payoffs are (10,10) if he sails past and (-10,20) if he dashes the ship into the rocks, where again the first number in each pair is how he values the outcome this morning, and the second is how he will value it this afternoon.

a) Write down the extensive form of this game. What action does Odysseus take this morning if he uses backward induction to make his decision? Explain.
b) Now add a new stage at the beginning of the game in which Odysseus decides whether or not to have his crew commit to lashing him to the mast (and plugging their ears) this afternoon. If he is lashed to the mast, he is rendered unable to select (T) in the afternoon. Otherwise the game is as above. Write down the extensive form of this new game and solve it by backward induction. Explain.

6. **Investment and Work Effort:** Suppose that Lisa must decide whether or not to invest \$40,000 in a new computer system for her company. If she buys the system, she must also hire a technician, Bart. The effect on the firm's revenues depends on Bart's effort. If Bart puts in high effort (H) in working with the new system, then revenues increase by \$120,000. If Bart puts in low effort (L) then revenues increase by only \$50,000. Bart has disutility of \$0 for low effort and \$10,000 for high effort.
    a. Suppose that Lisa pays Bart a salary of \$60,000 regardless of his effort. In this case, will Lisa make the investment?
    b. Suppose that Lisa writes a (legally binding) contract with Bart under which Bart gets a base salary of \$45,000 regardless of effort and a bonus of \$15,000 if he puts in high effort. Will Lisa now make the investment in this case? Would she if the cost of the investment was \$70,000?

7. **Entry Game:** Please go back to Assignment 1 and do #3 parts i-k.

8. **Tennis II:** Consider the tennis problem from Assignment 3 #2 part b. Suppose that the two players play this one volley for a \$1 million dollar prize. Suppose also that player 2 can decide to forfeit rather than play the volley in which case she gets \$300,000 and player 1 gets \$700,000.

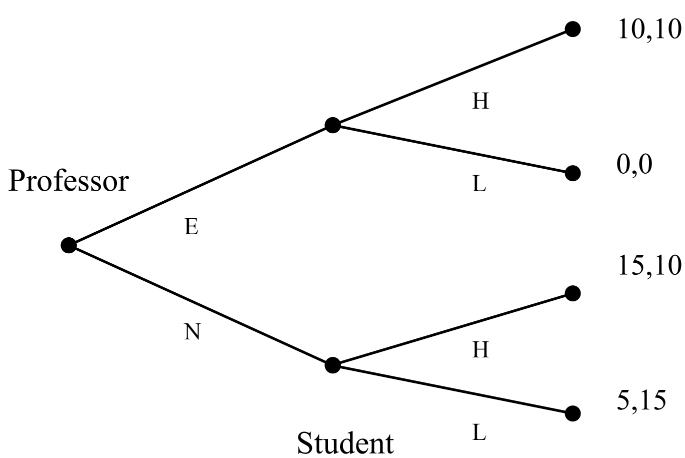
a) Write out the game tree for this game. If she cares only about her expected earnings, will player 2 forfeit or play? What are the SPNE of this game? Might your answer change if player 2 is risk averse?
b) Now suppose that player 2's utility function is $u(x)=x^{1/2}$ rather than $u(x)=x$ as above. Under this assumption, will player 2 now forfeit or play? Explain this result intuitively. Now what is the smallest (certain) $ payoff for forfeiting that would induce player 2 to forfeit? I.e., what is the *certainty equivalent* to the gamble of playing the volley.

9. **Trade Policy**: Suppose that Japan and the US are each deciding whether to keep their economies Open or Closed to trade (i.e., whether to take a more or less free trade vs. protectionist stance on trade policy). Suppose further that the payoffs are as follows:

		JA	
		Open	Closed
US	Open	4,3	3,4
	Closed	2,1	1,2

a. What is the Nash equilibrium of this game if the two countries make their decisions simultaneously?
b. Suppose now that play is sequential and Japan moves first. Find all pure strategy Nash equilibria of this game. Which are subgame perfect?
c. Would the US be better off moving first in this game? Are there other devices that you can think of that would help the US increase it's payoff?

10. **Should the professor give a final exam:** Consider the following problem faced by a professor and a student. The professor must decide at the beginning of the semester whether he will give a final exam (E) in the course or not (N). The student, after seeing the professor's choice, will then decide whether to put in high effort (H) or low effort (L) during the semester. Note that, according to the payoffs below, the professor would like the student to put in high effort, and the student would like the professor not to give an exam. Alas, what is to be done?



Write down the normal form representation of this game and find all Nash Equilibria in pure strategies. Explain intuitively why each is a NE. Which of these NE are subgame perfect? Explain intuitively why each NE is or is not an SPNE.

Sequential Bargaining

11. **Bargaining over a shrinking surplus:** Suppose that two players are negotiating over how to divide \$100. In each round one of the players makes an offer and the other can accept or reject that offer. If the offer is rejected, we go to the next round and \$30 disappears. Bargaining can last at most four rounds. (You can assume that the \$100 is infinitely divisible and that the players do not discount the future (i.e., each has discount factor $\delta=1$)).

    The exact sequence of events is as follows:
    Round 1: Player 1 offers player 2 a division of the \$100. Player 2 accepts or rejects. (move to the second round if player 2 rejects the offer)
    Round 2: Player 2 offers player 1 a division of the remaining \$70. Player 1 accepts or rejects. (move to the third round if player 1 rejects the offer)
    Round 3: Player 1 offers player 2 a division of the remaining \$40. Player 2 accepts or rejects. (move to the fourth round if player 2 rejects the offer)
    Round 4: Player 2 offers player 1 a division of the remaining \$10. Player 1 accepts or rejects. The game ends. (if the game gets to this round and player 1 rejects the offer, the players both leave with nothing)

    Use backward induction to determine how the bargain will be resolved (i.e., to find the outcome of the SPNE of the game). Specifically, how much will each player get, and how many rounds will the bargaining process actually last.

12. **Rubinstein Bargaining Model:** Consider a 4 period version of the Rubinstein Bargaining Model in which two players bargain over the division of a dollar.

    Players 1 and 2 take turns as follows. Assume that each round takes one time period:
    Round 1: Player 1 offers player 2 a division of the dollar, and then player 2 accepts (in which case the game is over) or rejects, in which case we move to round 2.
    Round 2: Player 2 offers player 1 a division of the dollar, and then player 1 accepts (in which case the game is over) or rejects, in which case we move to round 3.
    Round 3: Player 1 offers player 2 a division of the dollar, and then player 2 accepts (in which case the game is over) or rejects, in which case we move to round 4.
    Round 4: Player 2 offers player 1 a division of the dollar, and then player 1 accepts (in which case the game is over) or rejects, in which case the game is over and both players get 0.

    Assume that the dollar is infinitely divisible and that both players discount the future. Call the discount factor of player 1, δ_1, and the discount factor of player 2, δ_2. Assume that both discount factors are less than or equal to 1, but that they need not be the same.

Thus, for example, player 1 would value receiving \$1 two periods from now as equivalent to receiving \$ $\delta_1^2 \cdot 1$ today. If, for example, δ_1=0.5, then she would be indifferent between receiving \$1 two periods from now and \$0.25 today. Note that if δ_1=1 then player 1 is infinitely patient. If δ_1=0 then player 1 is infinitely impatient.

Use backward induction to show that at the unique SPNE of this game, player 1 gets $\$1-\delta_2+\delta_1\delta_2-\delta_1\delta_2^2$ and player 2 gets one minus that sum. In which round is the bargain settled?

13. **Rubinstein Bargaining Model:** Which player gets the larger payoff in the Rubinstein bargaining model? Explain intuitively. Is there a first mover advantage?

Finitely Repeated Games:

14. Consider the Prisoners' Dilemma. Confirm that it has a unique Nash equilibrium. Now consider a repeated game in which two players play the prisoners' dilemma (with simultaneous moves) once, then observe the outcome of this first stage, and then play a second time (again with simultaneous moves). What does the extensive form representation of this game look like (don't draw the entire tree)? How many (pure) strategies does each player have in her strategy set? Show that the Subgame Perfect Nash Equilibrium of this game is for each player to always defect.

15. **Conditionally Repeated Prisoners' Dilemma:** Consider a sequential game in which a Prisoners' Dilemma game is played in the first round (after which the outcomes are observed by both players). The game moves to a second round only if (C,C) was the outcome of the first round. Otherwise the game ends. Can the players cooperate in a SPNE of this game?

16. **Repetition of a Stage Game with Multiple NE:** Consider the following modified prisoners' dilemma game in which we add a third strategy of "partially" cooperating. This stage game has two nested prisoners' dilemmas, one in C and D, the other in C and P. It also has two pure strategy NE.

		Player 2		
		C	D	P
Player 1	C	10,10	0,12	5,11
	D	12,0	2,2	6,1
	P	11,5	1,6	8,8

Notice that C is strictly dominated by both D and P in this game. If this game is repeated once, can the players cooperate in a SPNE of the repeated game?