1. The Normal Form Representation

Basic Notation

n: number of players.
s_i: a (pure) strategy of player i.
$S_i = \{s_i^1, ..., s_i^m\}$: the strategy space (or strategy set) of player i. Here, player i has m strategies in her strategy space.
$s = (s_1, ..., s_n)$: the strategy profile of the n players; the "outcome" of the game.
$s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$: the strategy profile of the other $n-1$ players. Thus, we can write $s = (s_i, s_{-i})$ when that is convenient.
$u_i(s_i, s_{-i})$: the payoff to player i as a function of the strategy profile played by the n players in the game. Payoffs should be thought of as utilities of the outcomes, though we will occasionally assume for convenience that these are just the monetary outcomes.
S: the set of all possible strategy profiles.

Example: Rock Paper Scissors

$n = 2$.
$s_1 = R$: an example of a strategy played by player 1.
$S_1 = \{R, P, S\}$: the strategy space of player 1.
$s = (R, P)$: an example of a strategy profile played by the two players - player 1 plays Rock, player 2 plays Paper.
In that case $s_1 = R$ and $s_{-1} = P$. Suppose that each player values a win at 1, a loss at -1, and a draw at 0. Then the payoff $u_1(R, P) = -1$.
$S = \{(R, R), (R, P), ..., (S, S)\}$ is the set of all nine possible strategy profiles that could be played in this game - i.e., the set of all nine possible outcomes for the game. Notice that each of these strategy profiles corresponds to a cell in the 3x3 game matrix.

Mixed Strategies

$\sigma_i = (p_i^1, p_i^2, ..., p_i^m)$: a (mixed) strategy for player i is a probability distribution over the m pure strategies in player i's strategy set. Note that $\sum_j p_i^j = 1$. Note also that a pure strategy can be expressed as a mixed strategy that places probability 1 on a single pure strategy and probability 0 on each of the other pure strategies.[1]
The *support* of a mixed strategy is the set of pure strategies that are played with non-zero probability under the mixed strategy.
ΔS_i is the set of possible mixed strategies available to player i (i.e., the set of all probability distributions over the pure strategies of player i).
$\sigma = (\sigma_1, ..., \sigma_n)$: the (mixed) strategy profile of the n players.
$\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$: the strategy profile of the other $n-1$ players. Thus, as above, we can write $\sigma = (\sigma_i, \sigma_{-i})$ when that is convenient.
$u_i(\sigma_i, \sigma_{-i})$: the *expected payoff* to player i as a function of the mixed strategy profile played by the n players in the game. The outcome of the game is now random. We typically assume that players care a priori about the expected payoff, defined as the expected value of the utility of the outcome, where the probability of each possible outcome is determined by the mixed strategy profile being played.
Beliefs: Watson uses the same notational conventions for mixed strategies σ and beliefs θ. For example, player i's belief about the strategy profile being played by the other $n-1$ players is denoted θ_{-i} and is a probability distribution over the pure strategies of the other players.

[1] Note that Watson uses a bulkier but equivalent notation for mixed strategies. In Watson's notation $\sigma_i = (\sigma_i(s_i^1), \sigma_i(s_i^2), ..., \sigma_i(s_i^m))$. You are welcome to use either.

Domination

Dominated Strategy: For player i, a pure strategy s_i is (strictly) dominated if there exists another (pure or mixed) strategy $\sigma_i \in \Delta S_i$ for which $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

I.e., strategy s_i is dominated if player i has another strategy that gives her a higher expected payoff regardless of the strategy profile played by the other players.

Iterated Dominance: is the process of iteratively removing dominated (pure) strategies from the game. If this procedure yields a unique outcome (a unique pure strategy profile) then we can say that the game is solvable by iterated dominance.

Rationalizable Strategies: The pure strategies that survive iterated dominance are called the rationalizable strategies of the game. Similarly, the set of strategy profiles that survives iterated dominance is called the set of rationalizable strategy profiles of the game.

E.g., the only rationalizable strategy profile for Prisoner's Dilemma is (D,D), whereas the set of rationalizable strategy profiles for Chicken is {(C,C), (C,T), (T,C), (T,T)}, since both C and T are rationalizable for each player.

Nash Equilibrium

Best Response: For player i, a strategy σ_i is a best response to the strategy profile σ_{-i} if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$ for all $s'_i \in S_i$.

Note that σ_{-i} is a specific strategy profile that could be played by the other players in the game.

Since σ_i may not be the only best response to σ_{-i}, we will call $BR_i(\sigma_{-i})$ the set of best responses for player i to σ_{-i}, and note that $\sigma_i \in BR_i(\sigma_{-i})$.

We can also consider the set $BR(\theta_{-i})$ of best responses of player i to her belief θ_{-i} about the strategies being played by the other players.

Note that a (strictly) dominated strategy is never a best response.

Nash Equilibrium in Pure Strategies: A pure strategy profile $s^* = (s^*_1, ...s^*_n)$ is a Nash Equilibrium if each player's strategy is a best response to the strategy profile played by the other players in the game.

I.e., s^* is a Nash equilibrium if $s^*_i \in BR_i(s^*_{-i})$ for all players i,

or equivalently, if $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$ for all $s_i \in S_i$ and for all players i.

For example (R,P) is **not** a Nash Equilibrium of Rock, Paper, Scissors, since even though player 2 is playing a best response (Paper) to player 1's strategy (Rock), player 1 is not playing a best response to player 2's strategy (the best response to Paper is Scissors). Indeed, Rock, Paper, Scissors has no Nash Equilibrium in pure strategies.

Nash Equilibrium in Mixed Strategies: A mixed strategy profile $\sigma^* = (\sigma^*_1, ...\sigma^*_n)$ is a Nash Equilibrium if each player's strategy is a best response to the strategy profile played by the other players in the game.

I.e., σ^* is a Nash equilibrium if $\sigma^*_i \in BR_i(\sigma^*_{-i})$ for all players i.

So, a strategy profile is a NE if each player is playing a BR to the strategy profile played by the other players. Further, note that mixed strategies include pure strategies (i.e., we can always represent a pure strategy as a mixed strategy). We now have the following Theorem.

Nash's Theorem (1950): Every finite normal form game has at least one Nash Equilibrium in mixed strategies.

Let's reiterate that in Nash's Theorem, 'mixed strategies' is meant to include pure strategies. E.g., the unique NE of prisoner's dilemma can be expressed as $\sigma^* = ((0, 1), (0, 1))$. While not all finite strategic form games have NE in pure strategies (e.g., matching pennies, or rock paper, scissors), they all must have at least one NE (which may or may not be in pure strategies).

Example: Rock Paper Scissors:

$\sigma_1 = (\frac{1}{2}, \frac{1}{2}, 0)$: an example of a mixed strategy played by player 1. This mixed strategy plays R or P each with probability $\frac{1}{2}$ and S with probability 0 - i.e., it's equivalent to flipping a coin and playing Rock if heads and Paper if tails. The support of this mixed strategy is $\{R, P\}$.

$\sigma = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$: an example of a strategy profile played by the two players - each randomizes over R, P, and S with equal probabilities of $\frac{1}{3}$. This particular profile happens to be the unique NE of this game (recall that Rock Paper Scissors has no NE in pure strategies).

Suppose again that each player values a win at 1, a loss at -1, and a draw at 0. Then at the NE, each player has expected payoff $u_i(\sigma_i, \sigma_{-i}) = 0$ (note that the probability of each of the 9 possible outcomes under this mixed strategy profile is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$).

Indifference Property: A mixed strategy σ_i is a BR to σ_{-i} if and only if each pure strategy in the support of σ_i is a BR to σ_{-i}. Consequently:

1. Any mixed strategy over this support will be a BR to σ_{-i}, and
2. σ_{-i} makes player i indifferent to using each pure strategy in the support of σ_i.

Since at a NE, each player is playing a BR to the strategies used by the other players, then (2.) above in turn implies that at a NE, for any player i, σ^*_{-i} makes player i indifferent to using each pure strategy in the support of σ^*_i. We can often use this indifference to calculate the mixed strategies being played by various players at a NE.

2. The Extensive Form Representation

Definitions

A game of *complete information* is one in which the rules of the game, actions available to each player, and payoffs of each player are common knowledge. Thus, the entire game tree is common knowledge.

A game of *perfect information* is a game of complete information in which all information sets in the game tree are singletons. I.e., whenever a player is called upon to take an action, she knows exactly where she is in the tree, or equivalently, she knows the exact history of the game.

Backward Induction

Kuhn's Theorem: (1953) Every finite extensive form game with perfect information has at least one solution by backward induction.

Subgame Perfection

A *subgame* of an extensive form game is a node x and all successor nodes for which no node is in an information set that contains nodes that are not successors of x.

Subgame Perfect Nash Equilibrium: A NE is *subgame perfect* if, in the extensive form, it dictates a NE for each subgame of the game.

Not all NE are SPNE. At a NE that is not a SPNE, some player is playing a strategy that is a BR in the full game but is not a BR in some subgame. Note that this subgame must be *off of the equilibrium path* (in the game tree) of this non-SP NE. Thus, we can usually interpret a non-SP NE as a NE at which some player is making a *non-credible threat or promise* off of the equilibrium path.

In games of perfect information, solutions by backward induction correspond to SPNE. In games of complete but imperfect information, we can find SPNE by using a modified form of backward induction, using NE as solutions to subgames as needed whenever we run into non-trivial information sets.

Existence: Since every subgame of a finite game (of complete information) has a corresponding finite normal form, and every finite normal form game has at least one NE (in mixed strategies), every finite game (of complete information) must have at least one SPNE.

Thus, SPNE is a refinement of NE that has two desirable properties. First, SPNE are NE that do not involve incredible threats or promises. Second, every finite game (of complete information) has at least one SPNE.

Some Results for Finitely Repeated Games

If the stage game has a unique NE, then there is a unique SPNE of the finitely repeated game. The SPNE dictates that the players play the unique NE of the stage game in each round.

If the stage game has more than one NE, then there are more than one SPNE of the repeated game, and any sequence of Nash stage profiles can be supported by a SPNE of the repeated game. Further, while a Nash stage profile must be played in the last round of the repeated game, non-Nash play in early rounds *may* be supported by a SPNE.

Infinitely and Indefinitely Repeated Games

A *trigger strategy* is a conditional strategy that rewards other players for conforming to a particular behavior cycle and punishes them from deviating from it. Classic examples for the repeated prisoners' dilemma game are the *Grim trigger* and *Tit for Tat*.

Folk Theorem: (Friedman, 1971) Consider an infinitely repeated game in which the players discount future payoffs. Any behavior cycle that generates average payoffs per period $(u_1, ..., u_n)$ that are at least as great for each player as the payoffs $(u_1^*, ..., u_n^*)$ at some NE of the stage game can be supported by a SPNE of the infinitely repeated game, provided that the players' discount factors are sufficiently close to one.

Note that the same result can be applied to indefinitely repeated games (games in which the players do not know in which round the game will end), provided that players believe that the probability that the game will continue at each round is sufficiently close to one (i.e., that the players are sufficiently patient).

This result implies that there is a very large range of behavior that can be consistent with SPNE in some repeated games. These outcomes are supported by trigger strategies that are self enforcing at the SPNE.

3. Evolutionary Games

Single Population Game: In an n player game, players are randomly selected (n at a time) out of a single population of players to play the game.

Multiple Population Game: In an n player game, each of the n players is randomly selected from a unique population of players to play the game. E.g., in a two player game like chicken, player I is always chosen from population I and player II is always chosen from population II.

Fitness: The fitness F_σ of a strategy σ is the expected payoff to a player playing σ under random matching with other players. In a single population game this expected payoff depends on the relative proportions of individuals in the population playing various strategies. In a multiple population game, the fitness of a strategy being played in one population depends on the relative proportions of individuals in the *other* populations playing various strategies.

Evolutionary Dynamics: The proportions of players playing strategies with relatively high fitness (within their individual populations) are assumed to increase over time. This can be modeled in various ways, one example being the *replicator dynamics*.

Evolutionary Stable Strategy: In a single population game, a strategy σ is an ESS if, starting with a population of all σ players, a small group of 'mutants' playing any alternative strategy σ' would have lower expected payoffs (under random matching) than the σ' players. Thus, strategy σ is an ESS if for (arbitrarily) small values of ϵ and any alternative strategy σ',

$$(1-\epsilon)u(\sigma,\sigma) + \epsilon u(\sigma,\sigma') > (1-\epsilon)u(\sigma',\sigma) + \epsilon u(\sigma',\sigma')$$

ESS are stable under standard evolutionary dynamics (such as the replicator dynamics).

All ESS are NE strategies, but not all NE strategies are ESS. Therefore, evolutionary stability is a refinement of NE. For example, the game Chicken has 3 NE, two in pure strategies and one in mixed strategies. Of the three Nash strategies, only the mixed strategy is an ESS.

Whereas all finite games have NE, not all finite games have ESS. For example, the game Rock Paper Scissors has a unique NE in which both players use the mixed strategy (1/3,1/3,1/3). However, that strategy is not an ESS, and there is no ESS for this game. Similarly, under evolutionary dynamics, the population tends to cycle between different strategies (rock, paper, and scissors) rather than converging to the equilibrium at which 1/3 of the population plays each strategy.

4. Bayesian Games

The term *incomplete information* refers to one or more players not knowing the complete structure of the game. This often refers to not knowing the payoffs faced by some other player or players. We can say that some players don't know what *type* of player they are playing with, i.e., that player types are *private information.* A *Bayesian Game* is one in which players know the various exact types of players that might be in the game and have common prior beliefs about the probability distribution over the different types.

Bayesian Normal Form of a Bayesian Game is the normal form representation of the game in which Nature moves first to select player types according to the common priors.

Bayesian Nash Equilibrium: A NE of the Bayesian Normal Form game. This is also equivalent to the NE of the game construed as a game between the various possible types of players in the game. I.e., each possible type of player plays a BR given the strategies played by each of the other possible types of players.

Perfect Bayesian Equilibrium: A strategy profile (for the Bayesian Normal Form game) plus a set of beliefs (a probability distribution for each non-trivial information set in the game tree) such that:

1. sequential rationality: each player acts rationally at each information set given her beliefs at each information set
2. consistent beliefs: the beliefs at each information set are consistent with both (a.) the strategies being played and (b.) Bayes' rule wherever possible.

Note that Bayes' rule can not be used when we are conditioning on a zero probability event, and so 2b. will not restrict beliefs off of the equilibrium path. This has led to further refinements of PBE (such as 'equilibrium dominance' – also called the 'intuitive criterion' – and 'sequential equilibrium') that place greater restrictions on beliefs (i.e., on what set of beliefs we consider to be rational).