Assignment 5

1a. The payoff matrix is

	Employer	
		W_H	W_L
Worker	E_H	4,4	0,8
	E_L	6,-2	2,2

The NE is (E_L, W_L). Notice that since this is a simultaneous move game, each player's decision has no effect on the other's.

1b. Now the stage game is repeated. In any round, a player's move within the round still can not have any affect on her opponent's decision in that round. However, each player can select a *trigger strategy* under which her behavior in any round depends on the behavior of her opponent in past rounds. You can design such a strategy (e.g., the grim trigger or tit-for-tat) to reward cooperation and punish defection (by your opponent) today by making your future moves depend on your opponent's current move.

Note that $\delta = 0.8$. If the employer plays grim, then the expected payoffs for the worker to always cooperating or defecting (always giving high or low effort) are

$$\begin{aligned} V_{E_H} &= \sum_{t=0}^{\infty} \delta^t \cdot 4 \\ &= \frac{1}{1-\delta} \cdot 4 \\ &= 20 \end{aligned}$$

$$\begin{aligned} V_{E_L} &= 6 + \sum_{t=1}^{\infty} \delta^t \cdot 2 \\ &= 6 + \frac{\delta}{1-\delta} \cdot 2 \\ &= 14 \end{aligned}$$

So given $\delta = 0.8$, we have $V_{E_H} > V_{E_L}$. Note that for a sufficiently low discount factor ($\delta < 1/2$), this would not be the case.

1c. The payoff matrix is now:

	Employer		
		W_H	W_L
Worker	E_H	W_H-2 , 10-W_H	0,8
	E_L	W_H-0 , 4-W_H	2,2

And so we have:

$$\begin{aligned} V_{E_H} &= \sum_{t=0}^{\infty} (0.8)^t \cdot (W_H - 2) \\ &= 5 \cdot (W_H - 2) \end{aligned}$$

$$\begin{aligned} V_{E_L} &= (W_H - 0) + \sum_{t=1}^{\infty} (0.8)^t \cdot 2 \\ &= W_H + 8 \end{aligned}$$

So given $\delta = 0.8$, we have $V_{E_H} \geq V_{E_L}$ as long as $W_H \geq 4.50$.

3d. For the worker, grim is a BR to grim. To see this, note that, (1.)*as a response to grim*, playing grim is equivalent to playing E_H all the time, and that, (2.) when playing against grim, if we don't want to deviate from E_H today, we wont want to at any period in the future either (see class notes). We also need to make sure that (3.) the worker's threat to punish the firm forever if the firm cuts the wage at any time is credible - but that's true since (E_L, W_L) is a NE of the stage game, so both parties always punishing is a NE of any continuation subgame. Finally, we need to check that (4.) grim is a BR for the employer if the worker plays grim (you should verify that this is the case in this problem).

2a. The competitive price is \$20 per unit and the competitive quantity exchanged (here these are shares that are both bought and sold by dealers - who act as middle-men) is 50 units per period. Profits are zero. Note that this is a Bertrand game.

2b. Total profit in the market per day is the spread (\$0.50) times the quantity of shares exchanged: $\pi = \$0.50 \cdot 47.5 = \$23.75$. Thus the profit per trader is \$1.1875 per day.

2c. Against grim, the expected payoffs to cooperating or defecting all the time are:

$$\begin{aligned} V_C &= \sum_{t=0}^{\infty} \delta^t \cdot 1.1875 \\ &= \frac{1}{1-\delta} \cdot \$1.1875 \end{aligned}$$

$$\begin{aligned} V_D &= \$12.1875 + \sum_{t=1}^{\infty} \delta^t \cdot 0 \\ &= \$12.1875 \end{aligned}$$

So $V_C \geq V_D$ as long as $\delta \geq 0.902564$.

Note that dealers are getting a profit of about \$1.19 per day per dealer forever if all dealers cooperate while a defector would get an immediate payoff of about \$12.19 today and then nothing thereafter forever. Our result above is that cooperation is worthwhile to an individual dealer (playing against a grim trigger) as long as her discount factor δ is at least about 0.9 per day. Note that measured in days, this is a fairly small discount factor. It says for example, that getting a dollar 5 days from now is only worth about 60 cents to you today, since $0.9^5 \cdot \$1 \approx \$0.60$.[1]

Assignment 6:

1b. Both (M,M) and (HLPG,HLPG) are NE of the strategic form game. However, only HLPG is an ESS. M is not an ESS, since a small invasion of mutant HLPG players into a population of all M players will have higher fitness. They will get a payoff of 50 when they play M players but 60 when they play other HLPG players. Specifically, suppose that the proportion of players who are playing HLPG is some small number ϵ. Then

$$\begin{aligned} F_M &= (1-\epsilon) \cdot 50 + \epsilon \cdot 50 = 50 \\ F_{HLPG} &= (1-\epsilon) \cdot 50 + \epsilon \cdot 60 = 50 + \epsilon \cdot 10 \end{aligned}$$

So $F_M < F_{HLPG}$ for any $\epsilon > 0$.

On the other hand, HLPG is an ESS, since, for a population of HLPG players, an invasion of a small number of M players will be repelled, as $F_M < F_{HLPG}$ in that case as well. Let ϵ now be the proportion of players playing M:

$$\begin{aligned} F_M &= (\epsilon) \cdot 50 + (1-\epsilon) \cdot 50 = 50 \\ F_{HLPG} &= \epsilon \cdot 50 + (1-\epsilon) \cdot 60 = 60 - \epsilon \cdot 10 \end{aligned}$$

[1] Note also that the actual volume per day on the NASDAQ is much larger than in this example, and so the amount of profits actually generated were much (much) larger than in this example (tens of millions of dollars per day in total). Notice, however, that scaling up Q (and thus each payoff) by any factor has no effect on the critical level of δ in this problem.

We see that $F_M < F_{HLPG}$ for small ϵ.

2b. The payoff matrix for the five round game is:

There are two NE in pure strategies: (AD,AD) and (G,G). There are also two mixed strategy NE, ((0,3/4,1/4),(0,3/4,1/4)) and ((3/8,0,5/8),(3/8,0,5/8)), that we will ignore for now.

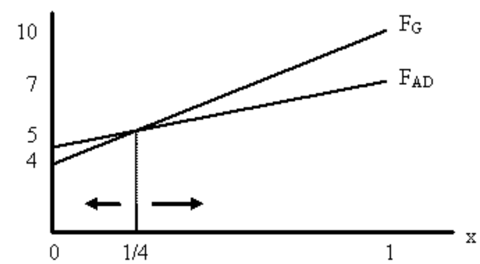
		II		
		AC	AD	G
I	AC	10,10	0,15	10,10
	AD	15,0	5,5	7,4
	G	10,10	4,7	10,10

2c.

If we call x the proportion of the population playing G, then

$$F_{AD} = x \cdot 7 + (1-x) \cdot 5 = 5 + 2 \cdot x$$
$$F_G = x \cdot 10 + (1-x) \cdot 4 = 4 + 6 \cdot x$$

So if $x = 0.2$, then $F_{AD} = 5.4 > 5.2 = F_G$. Consequently, x will fall toward zero over time. However, if we start with any level of $x > 1/4$, then $F_G > F_{AD}$ and so x will increase over time, converging to $x = 1$.



2d. Both AD and G are ESS.

2e. You can confirm that a small proportion of 'mutant' Grim players who invade a population made up entirely of σ' players will outperform the σ' players (under random matching). I.e., $(1-\epsilon)u(\sigma',\sigma') + \epsilon u(\sigma', G) < (1-\epsilon)u(G,\sigma') + \epsilon u(G,G)$. The same is true for an invasion of the σ' population by AD mutants. Consequently, σ' is not an ESS (either violation is sufficient to rule out σ' as an ESS).

2f. A BR to G is any strategy under which you play C for the first 4 rounds and then D in the 5th. E.g., play G for 1st 4 rounds and then (always) D in the fifth. Call that strategy nasty grim (NG). Noticed then, that with nasty grim (NG) in the strategy set, grim (G) is no longer an ESS. But then we can do better than NG as well, so as we add more sophisticated strategies in this finitely repeated game, we find cooperation unraveling just as with subgame perfection under full rationality (in the finitely repeated PD). Note also, however, that the full strategy set for the 5 times repeated PD is very (very) large (there are 256 decision nodes in the 5th round alone). So even in this simple game, we might expect real players to limit their attention to only a small subset of the master strategy set.

Note also that in a real game, we might expect cooperators (players of type either AC or G) to try to get around the random matching and refuse to play with defectors (AD, NG, etc.). If cooperators can wall themselves off from defectors effectively, then they can survive against any other strategy (this is discussed by Skyrms in Ch. 3). But then defectors would have an incentive to try to hide their types...

3. You will find that the dynamics drive all members of one population (either the type I or type II players) to play C (all of the time) and all members of the other population to play T all of the time. Thus we end up with one type of player learning to always be aggressive and the other to always be compliant. Payoffs are 3 for one population and 1 for the other. The average payoff over all players is 2, but the aggressive population is better off than the compliant one in equilibrium. Which of the two groups becomes the aggressive one, however, depends solely on the initial mixes in the two populations. There is no fundamental difference between the two populations, so the lock-in to one or the other outcome (i.e., which side is the winning or loosing side) is a function of idiosyncratic historical conditions. We can think of the outcome being a *convention* (like driving on the right or left, clocks being built to

run 'clockwise', etc.) that emerges and becomes supported by the historical reputations of the two populations.

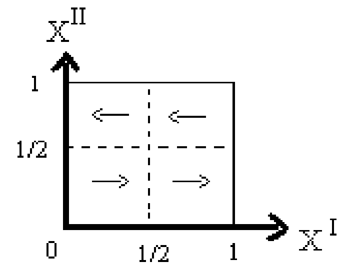
In terms of NE, we see that the two population evolutionary dynamics lead to one or the other pure strategy NE of the Chicken game.

Let's work through the picture for the dynamics. Let x^I be the proportion of members of population I who play tough (T). Let x^{II} be the proportion of members of population II who play tough (T). Then the fitness of each strategy (T and C) in population I depends on x_T^{II} as follows:

$$F_T^I = x^{II} \cdot 0 + (1 - x^{II}) \cdot 3 = 3 - 3 \cdot x^{II}$$
$$F_C^I = x^{II} \cdot 1 + (1 - x^{II}) \cdot 2 = 2 - 1 \cdot x^{II}$$

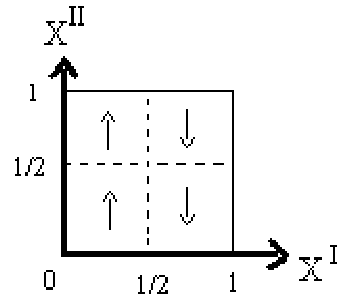
Then $F_T^I > F_C^I$ if $x^{II} < 1/2$. I.e., playing tough in population I gives a relatively high expected payoff if there are mostly chickens in population II.

This gives us the following dynamics for the mix in population I, i.e., for x^I:

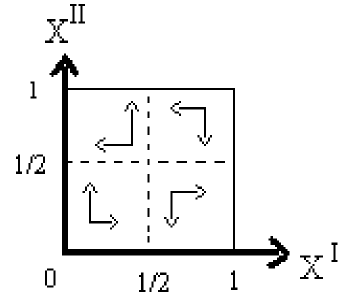


By the same kind of argument we will have fitnesses in population II depending on the mix of strategies played in population I, and $F_T^{II} > F_C^{II}$ if $x^I < 1/2$. I.e., playing tough in population II pays well if there are mostly chickens in population I.

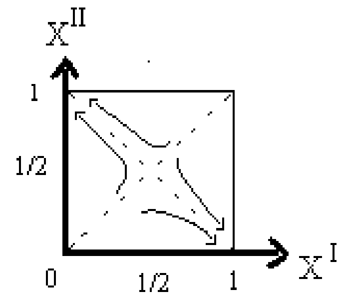
This gives us the following dynamics for the mix in population II, i.e., for x^{II}:



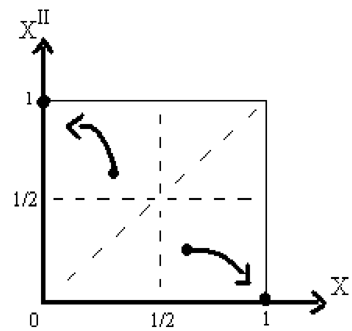
Putting these together, we have:



and thus,



So if we start for example with less than half of population I playing tough and more than half of population II playing tough, then chickens in population I do well and tough guys in population II do well. This reinforces the convention, and the tough players in population I learn to play chicken, while the chickens in population II learn to play tough. We are thus drawn to the upper left corner. If we had started closer to the lower right corner (with population I starting out relatively tough) then we would be drawn to that corner (i.e., that convention).



Finally, notice that this outcome is different than the outcome under a single population evolutionary game (see previous handout). In the single population version of the evolutionary chicken game, the ESS is a mixed strategy (and corresponds to the mixed strategy NE of the game). If players can only play pure strategies, then the single population equilibrium is polymorphic (some within the single population play C and others play T). However, since matching is random, some matches are T vs. T and C vs. C, lowering the expected payoff for each player to 1.5 in the single population game.