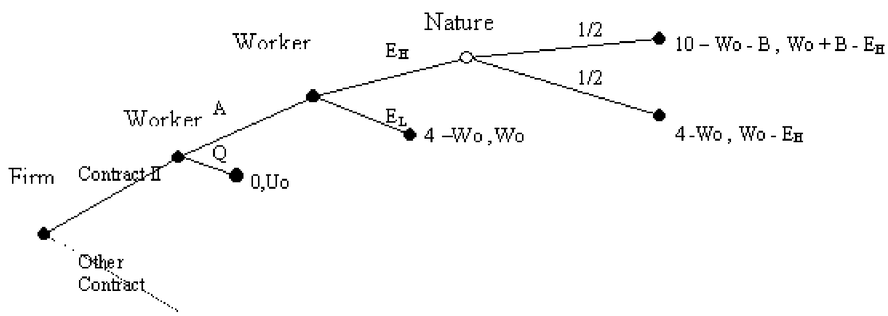
Economics 460
Chris Georges Some Help With Assignments 7 and 8

Assignment 7:

1a. Pay the worker \$5 if she exerts high effort and \$0 if she exerts low effort. This guarantees that she exerts high effort, and the firm's expected profit is \$2. If instead the firm paid \$3 for low effort, then the worker would be willing to exert low effort in which case the firm's profit would only be \$1.

1b. At a fixed wage, the worker selects low effort and the firm's profit is \$4-w. The lowest wage at which the worker will take the job, given that she will exert low effort, is \$3. Thus, the firm sets $w = 3$ and its profit is \$1.

1c. Here is the game tree.



To get a worker who has taken the job to select high effort rather than low effort, the firm would have to set the bonus high enough so that:

$$\frac{1}{2}(W_0 + B - E_H) + \frac{1}{2}(W_0 - E_H) \geq W_0$$

so that $B/2 \geq E_H$. This is called the *incentive compatibility constraint.*

To get this worker to take the job, given that she will expend high effort, the wage plus bonus must be high enough so that:

$$\frac{1}{2}(W_0 + B - E_H) + \frac{1}{2}(W_0 - E_H) \geq U_0$$

so that $W_0 + \frac{1}{2}B \geq U_0 + E_H$. This is called the *participation constraint.*

The firm will want to just meet the participation constraint, so we can replace the inequality with an equality, and so, given any bonus greater than or equal to $2E_H$, the wage can be calculated from the participation constraint. The firm's expecteed profit will then be

$$\pi = \frac{1}{2}(Y_H + Y_L) - (U_0 + E_H)$$

The firm will find this more profitable than setting $W_0 = 3$ and $B = 0$ and letting the worker select low effort (i.e., the fixed wage contract in part b.)if this profit level is greater than $Y_L - U_0$, which is true if $\frac{1}{2}(Y_H - Y_L) > E_H$ (as is the case in the current problem). Given the numbers in this problem, the firm's expected profit under the bonus contract is \$2 and under the wage-only contract is \$1.

The smallest bonus that is consistent with the incentive compatibility constraint (i.e., that will elicit high effort) in the current problem is $B = 4$. If the firm sets this bonus level, it will then want to set the base wage at $W_0 = 3$. The worker will be willing to work at high effort since the expected bonus ($2) is enough to compensate the worker for the disutility of high effort, and the expected total compensation ($5) minus the disutility of high effort ($2) is enough to compensate the worker for not quitting.

1d. The firm could set the base wage at 0, but would have to then offer a bonus of $10. Expected profit would be $2 as above.

1f. If the worker is risk averse, then the bonus contract will be risky to her. She will therefore require an additional risk premium (an additional base wage or bonus) to make the expected utility of choosing the riskier high effort greater than the certain utility of choosing low effort. To model this explicitly, we could posit that the worker has a concave utility function u() so that the incentive compatibility constraint becomes: $\frac{1}{2}u(w_0 + B - E_H) + \frac{1}{2}u(w_0 - E_H) \geq u(w_0)$. Watson provides an example with $u(x) = x^\alpha$, where $\alpha < 1$ corresponds to risk aversion(a concave utility function) and $\alpha = 1$ corresponds to risk neutrality (a linear utility function). Since, under risk aversion (concave utility function) the expected utility of a gamble is strictly less than the utility of the expected payoff from the gamble, the worker's compensation consistent with the incentive compatibility constraint will be higher if the worker is risk averse than if she is risk neutral. We can call the difference a risk premium.

1g. The minimum bonus to get the worker to work hard is now $8 (since the worker's incentive to give low effort is now higher under any given bonus), and the corresponding wage is still $3. Expected profit in that case is now $0.5, and expected profit under Contract I (the $3 wage-only contract at which the worker selects low effort) is still $1.00, so the firm should opt for the latter.

2. Consider the *incentive compatibility constraint*. For the worker to select high effort, we must have

$$W - E_H \geq p \cdot U_0 + (1 - p) \cdot W$$

so that the wage must satisfy

$$W \geq U_0 + \frac{E_H}{p}$$

I.e., the smallest wage the firm can set and still get high effort is $U_0 + E_H/p$. Note then, that as long as $p < 1$, (i.e., as long as monitoring is imperfect,) this wage is *strictly* greater than the opportunity costs of this work to the worker:

$$W > U_0 + E_H$$

Using the numbers from problem 1, the firm must pay a wage strictly greater than $5 to get high effort. For example, if $p = 1/2$, then the firm must pay a wage of at least $7 if it wants to elicit high effort. Intuitively, for the threat of being fired to be an effective incentive to work hard, the worker must have something to lose if fired. That something to lose is a higher wage than she can get elsewhere.

Extra: What if the cost of low effort was not zero (but was still less than the cost of high effort) so that we had $E_H > E_L > 0$). You should be able to show that this will reduce the efficiency wage in this problem, but not eliminate it (i.e., the wage that satisfies the incentive compatibility constraint will still be strictly greater than the wage that satisfies the participation constraint as long as monitoring is imperfect (i.e., as long as $p < 1$).

Assignment 8:

1a. For each case, draw the 3x3 game matrix. If both Players I and II are of type A, the NE is (Job 2, Job 2). If both Players I and II are of type B, the NE is (Job 1, Job 1). In each of these cases, the outcome is efficient (joint payoffs are maximized). Each worker selects her preferred strategy, and there is no externality when two workers of the same type team up in a job (since their productivities are identical, so neither is pulling down or pushing up the average).

Both Workers are Type A

II

I		Job 1	Job 2	Job 3
	Job 1	7 , 7	7 , 18	7 , 15
I	Job 2	18 , 7	18 , 18	18 , 15
	Job 3	15 , 7	15 , 18	15 , 15

I is Type A, II is Type B

II

		Job 1	Job 2	Job 3
	Job 1	6 , 6	7 , 4	7 , -10
I	Job 2	18 , 5	13 , 9	18 , -10
	Job 3	15 , 5	15 , 4	10 , -5

1b. When a Type A worker (Player 1) plays against a type B worker (Player 2), however, cell by cell inspection reveals there is no NE in pure strategies. Suppose for example that Player 1 starts off in Job 2 and Player 2 starts in Job 1 (as above, and as each would if she was the only player in the game). Then Player 2 now will want to switch to Job 2 to free-ride off of player 1's higher productivity (which raises Player 2's pay enough to make it worth the extra work effort). But then Player 1 will want to switch to Job 3 to get away from the slacker that is dragging down her pay. Player 2 won't follow Player 1 into Job 3, since it is involves too much work, so will then switch back to Job 1. But then, Player 1 has no reason to stay in Job 3 and will switch back to Job 2, ... and the best response cycle continues.

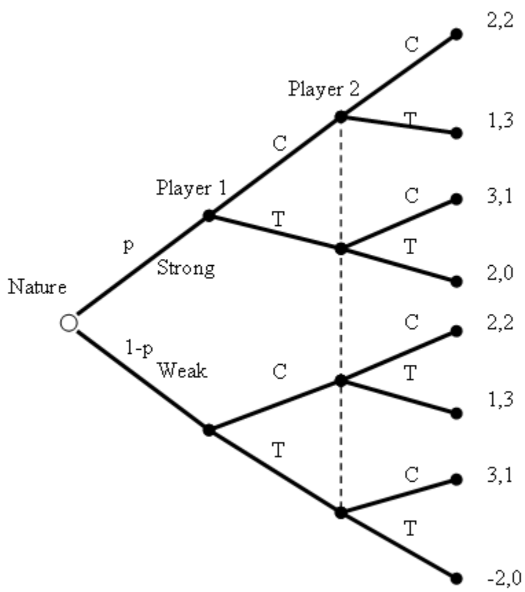
1c. We know, however, that there must be a NE in mixed strategies. Call the mixed strategy of player 1 $\sigma_1 = (p_1^1, p_1^2, p_1^3)$ where $p_1^3 = 1 - p_1^1 - p_1^2$. Call the mixed strategy of player 2 $\sigma_2 = (p_2^1, p_2^2, p_2^3)$ where $p_2^3 = 1 - p_2^1 - p_2^2$. Then note that Job 1 is strictly dominated for player 1 and Job 3 is strictly dominated for player 2. Thus it must be that $p_1^1 = 0$ and $p_2^3 = 0$. Thus we only need to consider the indifference conditions over two strategies for each player to find that the NE is σ =((0,1/5,4/5),(2/5,3/5,0)). Player 1 (who is of type A) will most typically select Job 3, but will sometimes select Job 2. Player 2 (who is of type B) will most likely select Job 1, but will sometimes select Job 2. The mixed strategies keep each player guessing about what the other will do enough to get us out of the BR cycle above. Note, however, that player 1 has to place high probability on taking Job 3 in order to discourage player 2 from taking Job 2 all of the time. Thus, player 1's expected utility is lower in this case than it is in the first case. She is in a Rat Race in which her utility is lower both because she often takes the excessively hard Job 3 and because when she doesn't take Job 3, she sometimes has to share Job 2 with the slacker who lowers her income.

Question: would the slacker prefer to play against a go-getter or against another slacker? (You might be surprised by the answer.)

1d. Finally, notice that if only Job 1 and Job 2 were available, type A workers would not be able to get away from type B workers. I.e., the establishment by the firm of the Job 3 job type is a mechanism which allows the workers to at least partially sort by type (a separating equilibrium) and also elicits excessively high effort from Type A workers in equilibrium. While we have not modeled it here, this gives us some sense that it may be in the interest of the firm to establish job categories or other mechanisms such as promotion tournaments that induce rat races.[1]

2. Player II does not know whether Player I is strong or weak. There is a common prior belief is that I is strong with probability p and weak with probability (1-p). If I is strong, then the payoffs to (T,T) are (2,0). If I is weak, then the payoffs to (T,T) are (-2,0). Otherwise, the game payoffs are the standard Chicken payoffs.

The extensive form representation for this Bayesian game of Chicken has nature moving first to select Player I's type, and the Bayesian normal form of this game is just the normal form representation of that extensive form game:



		II	
		C	T
I	CC	2,2	1,3
	CT	3-p,1+p	3p-2,3p
	TC	2+p,2-p	1+p,3-3p
	TT	3,1	4p-2,0

There is clearly a pooling BNE = (TT,C) for any common prior belief p. If $p \leq 1/2$ then a separating BNE (TC,T) also exists. However, if II believes the likelihood of I being strong is greater than 1/2, then this separating equilibrium does not exist and II will never play T in equilibrium.

3. Though there are only two firms, consider each of the three *types* of firm (Firm I, low cost type Firm II, high cost type Firm II) separately. This is a useful approach when the strategy sets are continuous (since we can't write out a game matrix).

Low Cost Firm II: If Firm II has low cost, then it faces demand $P = 540 - q_1 - q_2^L$. Thus it's profit maximizing quantity to produce is $q_2^L = (460 - q_1)/2$. I.e., this is its best response to Firm I's output q_1.

High Cost Firm II: By the same argument, a high cost type Firm II's best response to q_1 is $q_2^H = (440 - q_1)/2$.

Firm I: Firm I doesn't know which type of Firm II it will face, so perceives its expected demand to be $P = (1/2)(540 - q_2^L - q_1) + (1/2)(540 - q_2^H - q_1)$. Calling $\bar{q}_2 \equiv (q_2^L + q_2^H)/2$ we can write this

[1] I.e., the game of complete information that we have been working with in this problem could be thought of as being nested within a broader game of incomplete information between the firm and the workers, where the firm doesn't know the workers' types but may look for strategies that will induce a separating equilibrium.

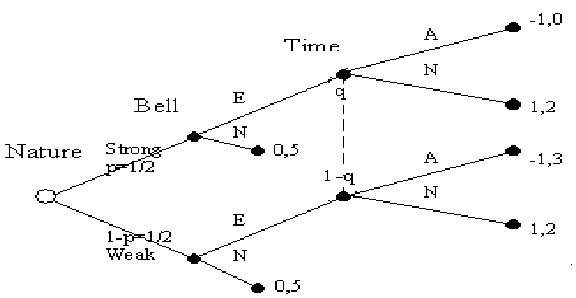
more compactly as $P = 540 - \bar{q}_2 - q_1$. Thus its best response to the expected mix of the two types is $q_1 = (450 - \bar{q}_2)/2$.

Solve the three best responses simultaneously to get the BNE quantities $q_1 = 150$, $q_2^L = 155$, $q_2^H = 145$. Thus, if Firm II turns out to have low cost, then the market price will be \$235, whereas if Firm II turns out to have high cost, the market price will be \$245.

Firm I prefers in this situation (with incomplete information) to face the high cost type of Firm II, since then Firm II has lower output and so the market price is higher. However, if Firm I *knew* with certainty which type firm it would face, it would act differently. E.g., if Firm I knew it would face a high cost Firm II, then it would play a BR to that firm's lower quantity and so its equilibrium quantity would be $q_1 = 153.33$ rather than 150.

4. See class notes.

5a. The game tree and Bayesian normal form of this version of the game are:



Time

Bell		A	N
	EE	-1 , 3/2	**1 , 2**
	EN	-1/2 , 5/2	1/2 , 7/2
	NE	-1/2 , 4	1/2 , 7/2
	NN	**0 , 5**	0 , 5

First notice that there are two BNE (Bayes Nash Equilibria) in pure strategies. BNE are NE of the Bayesian normal form of the game (which uses the un-updated prior probability $p = 1/2$ to calculate expected payoffs). The (pure strategy) BNE of this game are (EE,N) and (NN,A). Both are pooling equilibria (equilibria in which both Bell types take the same action). Notice that (EN,N) is not a BNE, since, at this strategy profile, while Time is playing a BR to Bell's strategy, Bell is not playing a BR to Time's strategy. Bell's BR to N is EE (if Time won't advertise, Bell should enter whether it is strong or weak).

Now let's look at PBE (Perfect Bayesian Equilibrium). PBE is a refinement of BNE, so we are asking whether the two BNE above survive this refinement. I.e., are there PBE that correspond to these BNE?

First, consider the strategy and beliefs profile (EE,N,q=1/2). For this to be a PBE, the strategies of each player must be sequentially rational (rational at each information set given the beliefs at each information set), and the beliefs must be consistent with the strategies being played and (if applicable) Bayes Rule.

Consider the belief $q = 1/2$ at Time's information set. Note that Bell's strategy induces the belief by Time that $q = 1/2$. I.e., if Bell announces that it will use strategy EE, and Time believes it, then Time has no direct reason to update the prior probability (since EE provides no direct signal about Bell's type). This belief is also consistent with Bayes' Rule: given the strategy EE, we have $q = \text{Prob}[\text{strong}|E] = \text{Prob}[\text{strong}] \cdot \text{Prob}[E|\text{strong}]/\text{Prob}[E] = (1/2) \cdot 1/1 = 1/2$.

Now consider sequential rationality. Given Time's belief when it reaches information set I_1 that Bell is strong with probability $q = 1/2$, its rational (expected payoff maximizing) move at this information set is N (since $(1/2) \cdot 3 < 2$). Expecting Time to not advertise if Bell enters, Bell should enter regardless of its type, so E and E are rational at Bell's two decision nodes (information sets). Thus, (EE,N,q=1/2) is a PBE.

Why is (EN,N,q=1) not a PBE? This should not be a surprise, since (EN,N) is not a BNE of this game (since EN is not a BR to N). To see explicitly that it is not a PBE, first consider the belief $q = 1$. Bell's strategy EN does induce belief $q = 1$ for Time, as it requires Bell to enter only if it is strong, making entry a signal that Bell is strong. Further, with belief $q = 1$, Time should select N at its information set I_1 (i.e., if Bell enters, Time should not advertise, since entry is a signal that Bell is strong). However, knowing that Time will select N, both Bell types should enter. Thus the strategy EN violates sequential rationality for Bell. A strong Bell will want to enter, given Time's belief that entry signals strength ($q = 1$), but a weak Bell will want to exploit Time's belief in the signal and enter as well.

Extra: Now consider the other BNE, (NN,A). In the original complete information game, (N,A) was a NE but was not SP, since it involved the non-credible threat that Time would advertise if Bell entered. Now in this incomplete information game, there is the possibility that Bell is weak and so that Time will indeed want to advertise after Bell enters. Time will thus advertise, after Bell enters, if it has a low enough belief q that Bell is strong. Bell's strategy NN does not place any restrictions on Times belief at I_1, which is off the equilibrium path (i.e., NN does not restrict Time's belief about Bell's type if Bell surprised Time and entered). Further, Bayes' Rule can't be used here, since the probability of Bell entering is 0 according to Bell's strategy. Thus, any belief q is admissible (consistent with Bell's strategy). Note then that as long as Time believes that $q \leq 1/3$, then it should select A if Bell enters. Thus for any belief $q \in [0, 1/3]$, we have that (NN,A,q) is a PBE. Notice that if Time is pessimistic and believes that entry by Bell indicates that there is greater than a 33 $\frac{1}{3}$% chance that Bell is strong, then q will be greater than 1/3, and this pooling PBE will not exist.

Extra: Do we really think that Time should hold a belief $q \leq 1/3$ if Bell plays NN? Notice that both types of Bell prefer the other BNE, so the "intuitive criterion" does not help us here either. I.e., it is hard to construct an argument that, if Bell entered in this case, it should be interpreted as a signal of strength.

Extra: Finally, consider what would happen if we reduced the payoff 3 for Time to 1. Now we are back to a situation in which Time should never advertise after Bell enters (A is dominated at I_1). Indeed, though (NN,A) is still a BNE of the game in this case (you can check the game matrix), there is no longer a corresponding PBE. There is no belief $q \in [0, 1]$ for which it would be rational for Time to carry out A at information set I_1. Note that (NN,A) is actually still a SPNE, since there are no proper subgames of the game due to the incomplete information. However, while it does not technically violate subgame perfection, it certainly violates the spirit of backward induction. Technically, it violates sequential rationality. This is an example of a general result: as a refinement of BNE, PBE rules out any action at an information set off the equilibrium path that is strictly dominated at that information set, as there is no believe that can support such an action.

5b. You should write out the 4x4 Bayesian strategic form game matrix to see that there are two pure strategy BNE of this Bayesian game:

i. (NN,EE) is a pooling equilibrium.

ii. (NA,EN) is a separating equilibrium.

where I am listing player I as Time and player II as Bell, and the first element of Time's strategy indicates what Time does if it is passive and the first element of Bell's strategy indicates what Bell does if it finds itself at the right hand information set. Thus, for example (NA,EN) indicates that, for example if Time is tough, then it chooses to make an aggressive (A) signal.

Each of these equilibria is also supported by Perfect Bayesian Equilibria. E.g., there are consistent beliefs q_1 and q_2 for which (NN,EE,q_1, q_2) is a PBE. Similarly, there are consistent beliefs q_1 and q_2 for which (NA,EN,q_1, q_2) is a PBE. You should solve for these beliefs and explain why these strategy and belief profiles are PBE.