Economics 285
Chris Georges

Help With Practice Problems 3

Chapter 8:

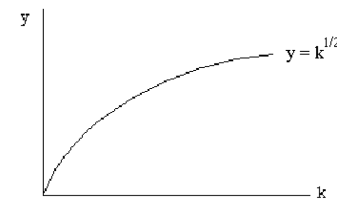
1. Questions For Review 1,4: Please see text or lecture notes.

2.

A note about notation: Mankiw defines k slightly differently in Chs. 8 and 9. In Ch. 8, where there is no technological progress, k is capital per worker ($k = \frac{K}{L}$). In Ch. 9, when we add technological progress, k is capital per effective worker ($k = \frac{K}{L \cdot E}$). Note that the latter definition is more general and is consistent with the first definition, since we will assume that, when there is no technological progress, $E = 1$ at all times. Indeed, the model in Ch. 8 is just the model in Ch. 9 with $E = 1$.

2.a. The per-worker production function in this problem is $y = k^{1/2}$. To get this, divide the production function by the number of workers L:

$$y = \frac{Y}{L} = \frac{F(K,L)}{L} = \frac{K^{1/2}L^{1/2}}{L} = (\frac{K}{L})^{1/2} = k^{1/2}$$



Note that this is a Cobb-Douglas production function ($F(K,L) = AK^{\alpha}L^{1-\alpha}$) with $A = 1$ and $\alpha = 1/2$. This production function displays diminishing returns to capital (diminishing MPK), and so its per-worker representation is concave.[1]

2.b. The basic equation for the Solow growth model is the equation describing how the capital stock evolves. In the absence of technological change, this equation is

$$\Delta k = s \cdot f(k) - (\delta + n) \cdot k$$

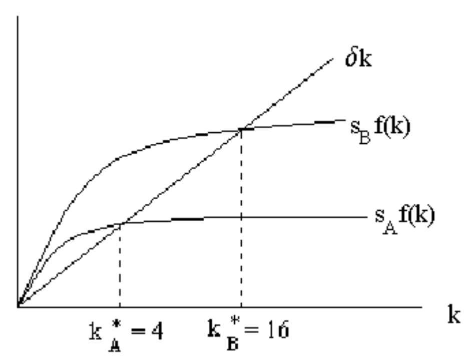
In part **b**, we have no population growth, so $n = 0$:

$$\Delta k = s \cdot f(k) - \delta \cdot k$$

This is the simplest version of the model.

In the *very long run*, after k (capital per worker) has adjusted to its steady state equilibrium k^*, it no longer grows or shrinks over time ($\Delta k = 0$). Saving (investment) out of current income is then just sufficient to replace the capital that depreciates each year. Calculate k^* by setting $\Delta k = 0$ in the last equation:

$$\begin{aligned} s \cdot f(k) &= \delta \cdot k \\ s \cdot k^{1/2} &= \delta \cdot k \\ k &= (s/\delta)^2 \end{aligned}$$



[1] The same is true for any Cobb-Douglas function with $\alpha \in (0,1)$.

Thus, if $\delta = 0.05$, country A with saving rate $s = 0.1$ has steady state capital stock $k^* = 4$, and country B with saving rate $s = 0.2$ has steady state capital stock $k^* = 16$.[2]

Plugging these levels of k^* back into the per worker production function, we get steady state per worker incomes of $y^* = 2$ and $y^* = 4$ in countries A and B respectively.[3] With twice the saving rate, country B ends up (in this problem) with twice the output per worker in the very long run.

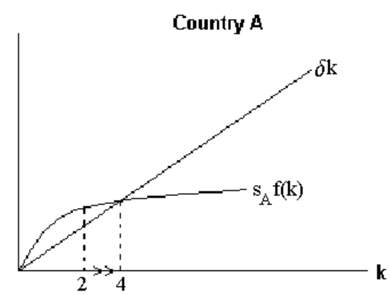
Finally, we can use the saving rates s in the two economies to get steady state per worker consumption $c^* = (1-s) \cdot y^*$ of 1.8 and 3.2 respectively. If we take per capita consumption to be a measure of the *standard of living* in each country, then this says that country B's standard of living will be about 75 percent greater than country A's in the very long run (78 percent to be precise: $c_B^*/c_A^* = 1.78$).

Note that national saving rates in Japan and the U.S. have averaged roughly 20 and 10 percent respectively of national income in recent history. The Solow model with the production function given in this problem thus predicts that the average standard of living in Japan should be about 75 percent greater than that in the U.S. in the very long run. However, this prediction depends on the production function assumed. Most estimates suggest that $f(k) = k^{1/3}$ would be closer to the fact (i.e., capital is less (and labor more) productive than we assumed above). Using this production function, we would find that the Solow model predicts the very long run Japanese standard of living to be about 25 percent higher than that in the U.S (with capital less productive, a high rate of saving and investment gives a country a more modest advantage).[4]

2.c.

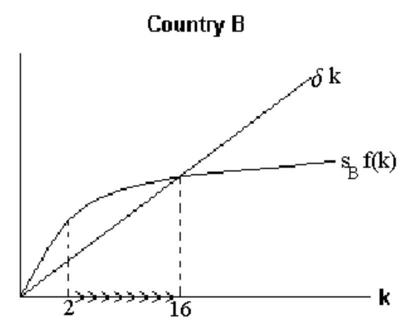
Country A:

Year	k	$y = k^{1/2}$	$c = (1-s)y$	$i = sy$	δk	$\Delta k = i - \delta k$
0	2	1.414	1.273	0.141	0.100	0.041
1	2.041	1.429	1.286	0.143	0.102	0.041
2	2.082	1.443	1.299	0.144	0.104	0.040



Country B:

Year	k	$y = k^{1/2}$	$c = (1-s)y$	$i = sy$	δk	$\Delta k = i - \delta$
0	2	1.414	1.131	0.283	0.100	0.183
1	2.183	1.477	1.182	0.295	0.109	0.186
2	2.369	1.539	1.231	0.308	0.118	0.190

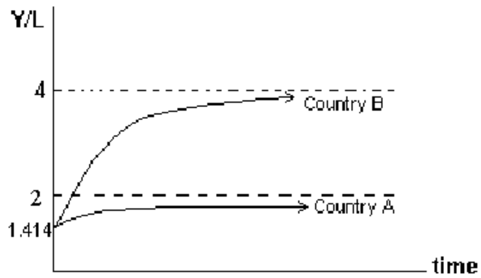


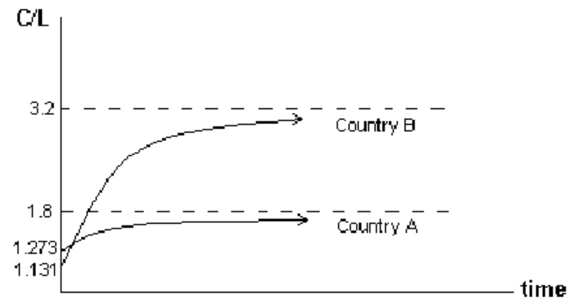
[2] Note: please **DO NOT** memorize the formula $k^* = (s/\delta)^2$ as it is only correct for the specific production function in this problem. Rather, you should learn the method for solving for k^*, which is to set $\Delta k = 0$.

[3] Equivalently, note that $y^* = k^{*1/2} = s/\delta$ in this particular problem.

[4] China's saving rate has been about 0.4 (40% of GDP) in recent years. If this persists, and the production function is $f(k) = k^{1/2}$, then the Solow model would predict that China's standard of living would be 166% greater than that of the U.S. in very long run equilibrium. Similarly, if the production function was $f(k) = k^{1/3}$, then China's standard of living would be 33% higher than that of the U.S. in very long run equilibrium.

Both countries start with the same amount of capital per worker, and thus output per worker. However, country B consumes less and saves more of its income than does country A. Specifically, country B consumes 80% of its income whereas country A consumes 90% of its income. Thus, country A starts off with higher consumption per worker (a higher standard of living) than country B.

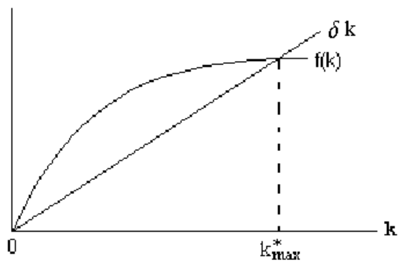


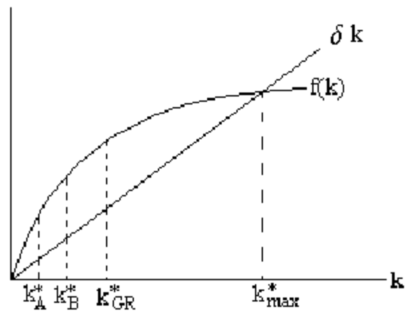


However, since country A saves less than country B, it also is investing less. 20% of country B's output is made up of investment goods (new plant and equipment) which will be added to the following year's capital stock whereas only 10% of country A's output is investment goods. Consequently, country B will end up with more capital per worker and thus output per worker than country A in the very long run. We saw above (in part **c**) that the higher saving rate in country B induces enough extra output per worker y^* in the very long run so that consumption per worker is also higher (78 percent higher) in the very long run.

If we had continued with the calculations for the next several years, we would have found that country B's consumption per worker c overtakes country A's in the fifth year. On the way to their respective steady state equilibria, consumption per worker grows faster in country B than in country A, since country B is saving and investing more. However, this growth is only transitory for both countries. In the very long run, saving alone is insufficient to cause sustained growth because of diminishing returns to capital. As each country adds capital stock to its economy, output and thus saving do not rise as quickly as depreciation, and so each economy reaches a steady state at which output per worker and standards of living (consumption per worker) no longer grow. At these steady states, the output per worker and standard of living of country B are greater than those of country A by 100 percent and 78 percent respectively.

2.d. Notice that the Golden Rule saving rate (the rate that maximizes consumption per worker in very long run equilibrium) must be greater than 10%, since we know that country A could increase it's very long run consumption per worker by raising its saving rate. We also know, however, that if the saving rate is increased above the Golden Rule rate, consumption per worker will decline in the very long run. So what is this Golden Rule saving rate? Is country B also below it? One way to answer this question is to experiment with different saving rates. Another is to consider the following graph.





Any level of k between 0 and $k^*_{\max}$ is a very long run equilibrium k^* for some saving rate s. Further, in very long run equilibrium, saving just offsets depreciation, so the gap between output and depreciation is also the gap between output and saving. Consequently, consumption per worker is given by the gap between the two curves in the graph (output and depreciation per worker). The gap between the curves is greatest where the slopes are the same. Call this capital stock k^*_{GR}. Now country B has a saving rate of 20% ($s = .2$) which gives it an equilibrium per worker capital stock k^* of 16. The slope of the of the production function (the MPK) at $k = 16$ is 1/8 or 0.125. Thus the production function is steeper here than is the depreciation line, and so k^*_{GR} must be greater than 16. Consequently, the golden rule saving rate must be greater than .2. Raising the saving rate a bit further would increase equilibrium output by more than equilibrium saving and so increase equilibrium consumption per worker.

You can solve explicitly for the golden rule level of s explicitly in the current problem by setting the MPK equal to the depreciation rate (to get $k^*_{\mathrm{GR}} = 100$) and then using $k^* = (s/\delta)^2$ to get $s_{\mathrm{GR}} = 0.5$.[5] [6]

Now, most estimates suggest that, as with country A in our example, the national saving rate in the U.S. is substantially below the golden rule rate. Thus, we could raise our standard of living in the very long run by raising our saving rate. Do we want to do this? Notice that to do so would be painful in the short run. To increase saving today means cutting consumption today (today's income is fixed — it depends on past investment) and for the next several years. The payoff would be higher consumption per worker in the very long run.

The model highlights a conflict between the interests of current and future generations, which is faced by economies operating below the Golden Rule. The less we save now, the more we can consume now, but the less we will be able to consume in the future. In order to raise standards of living in the future, we have to lower our standard of living in the present.

If we did want to raise the national saving rate, how would we want to do this? Since saving is done by households, businesses and government, public policy might be aimed at any or all of these groups. For example, the government can raise it's saving directly by raising taxes or cutting its spending. It can also encourage private saving through the tax code. For example, some politicians and economists support various flat tax and consumption tax proposals in part on the grounds that these tax codes might promote more saving in the economy. Similarly, the recent large cut in the

[5] Recall that $k^* = (s/\delta)^2$ is true only for the current problem – so more generally, once you have k^*_{GR} you want to find the saving rate s that yields this level of k as a steady state – i.e. the level of s that makes $\Delta k = 0$ at k^*_{GR}.

[6] Note also that the more general condition for the Golden Rule when there is population growth and/or technological progress is $MPK = \delta + n + g$.

corporate tax rate in the U.S. was passed with the hope that it would increase corporate saving and investment spending.

It should be cautioned, however, that while national income accounting treats government spending as a drain on national saving (i.e., as a form of consumption rather than investment), this is somewhat misleading. Highways, sewers, education, etc., are part of the social capital stock of the economy and government spending on them should be considered to be productive investment. In practice, cutting the deficit by cutting government capital outlays will not improve very long run standards of living unless private investment is more productive than public investment.

2.e. We saw above that the highest rate of consumption per worker possible in very long run equilibrium is attained at $s = .5$. This section of the problem simply asks us to try out various saving rates to see what the actual levels of c^* are for each saving rate.

Here is one way to proceed. We saw above that, *in this particular problem*, the steady state level of capital per worker k is:

$$k^* = (s/\delta)^2$$

Notice then that the steady state level of output per worker y is (again, in this particular case)

$$\begin{aligned} y^* &= f(k^*) \\ &= s/\delta \end{aligned}$$

and the steady state level of consumption per worker c is

$$\begin{aligned} c^* &= (1-s) \cdot y^* \\ &= \frac{(1-s) \cdot s}{\delta} \end{aligned}$$

Then, plugging in $\delta = 0.05$ and the various levels of the saving rate s that we want to consider, we have:

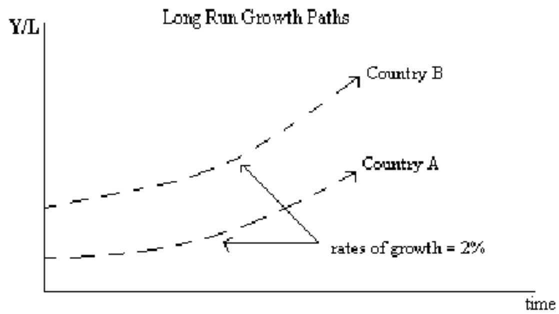
s	k^*	y^*	c^*
0	0	0	0
.25	25	5	3.75
.50	100	10	5
.75	225	15	3.75
1	400	20	0

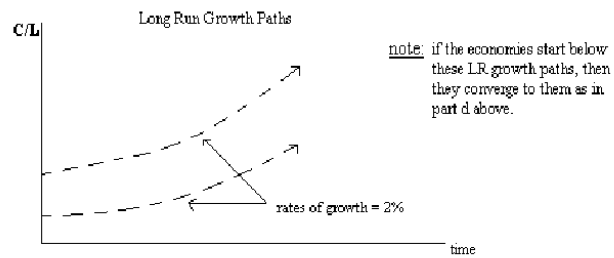
Notice that the steady state levels of both the capital stock per worker and output per worker are monotonically increasing in the saving rate. However, the same is not true of steady state consumption per worker, which is the highest at an intermediate saving rate: here $s = 0.5$ (we saw above that this is indeed s_{GR} in this example). A high saving rate raises output and thus incomes, but lowers the fraction of this income that we consume rather than save. If the saving rate is too high, we are producing a lot of income, but saving so much of it that consumption per worker is less than its maximum sustainable value.

2.f. In the Solow model in very long run equilibrium, output Y in a country grows at rate $n + g$ (the rate of population growth in that country plus the rate of technological progress), and output per worker Y/L grows at rate g (the rate of technological progress). Since output per worker grows at rate g, so does consumption per worker (the standard of living). Notice that these growth rates are independent of the saving rate.

Thus, in the current question, output grows at a rate of 3% per year in each country, output per worker grows at a rate of 2% per year in each country, and consumption per worker grows at a rate of 2% per year in each country, in the very long run.

The two countries will, however, have permanent differences in their *levels* of output and consumption per worker because of the difference in their saving rates. Since the growth rates are the same, the ratios of these variables for the two countries will be stable. Indeed, in the present case, these ratios turn out to be the same as in part **c**, as we will see below.





2.g. When there is technological progress we have the full equation of motion for k:

$$\Delta k = s \cdot f(k) - (\delta + n + g) \cdot k$$

where $k = \frac{K}{L \cdot E}$. E is the efficiency of labor, which is growing over time at rate $g = 0.02$. For consistency with the rest of the problem above, assume that the production function is now $Y = K^{1/2}(L \cdot E)^{1/2}$. Then we still have $f(k) = k^{1/2}$.

Now, solving the equation of motion above for $\Delta k = 0$, in terms of the parameters of the model, we find that $k^* = (\frac{s}{\delta+n+g})^2$. Thus, we can also calculate $y^* = k^{*1/2} = \frac{s}{\delta+n+g}$ and $c^* = (1-s) \cdot y^* = (1-s) \cdot (\frac{s}{\delta+n+g})$.[7] Thus, for country A we have $k_A^* = 1.5625$, $y_A^* = 1.25$ and $c_A^* = 1.125$. For country B we have $k_B^* = 6.25$, $y_B^* = 2.5$ and $c_B^* = 2.0$. So the ratio of consumption per effective worker in countries B and A is $2.0/1.125 = 1.78$.

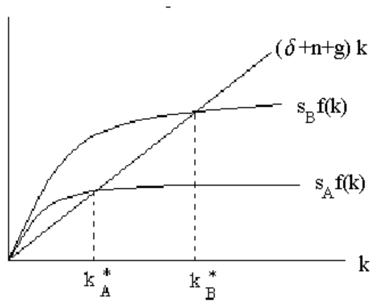
Now what about the standard of living, C/L? Since in steady state equilibrium we have $\frac{C}{L \cdot E} = c^*$, then we also must have $C/L = c^* \cdot E$, which is growing over time at rate g. Assuming that the paths of technology (E) are the same in both countries, then the ratio $\frac{(C/L)_B}{(C/L)_A}$ is c_B^*/c_A^*, which is 1.78.

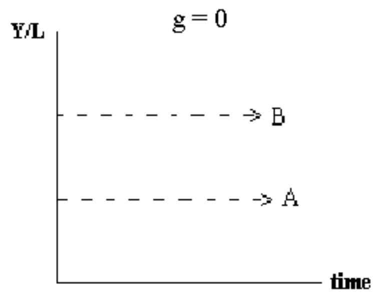
Notice that the same logic applies to the path of output per worker in the two countries. Since for any country, $\frac{Y}{L \cdot E} = y^*$, then $Y/L = y^* \cdot E$, which is growing at rate g. The ratio $\frac{(Y/L)_B}{(Y/L)_A}$ is thus y_B^*/y_A^*, which is 2.

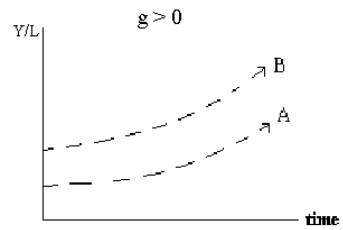
Thus, just as before, country B's output per worker is twice as great as country A's in the very long run, and its consumption per worker is 78 percent greater. Both output per worker and the standard of living are now growing over time (at rate g) in each country, but country B's output per worker will be consistently twice that of country A's, and its standard of living will be consistently 78% higher than country A's, on these very long run equilibrium paths. These are precisely the paths that are pictured in the diagram above.

[7] Again these formulas hold only for this specific example.

How general is this result? In general, the equilibrium of the Solow model with technological progress ($g > 0$) is *qualitatively* similar to the equilibrium of the model without technological progress ($g = 0$) but with the steady state equilibrium levels of per-worker variables (e.g., K/L and Y/L) growing at rate g. So for example, a country with a higher saving rate will necessarily have a higher level of the steady state growth path of output per worker than will a country with a lower saving rate, while the countries will have the same growth rates along these paths.

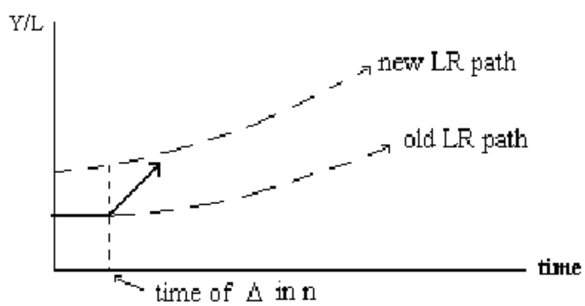






However, the consumption and output ratios above will not always be *numerically* identical in the two cases ($g = 0$ and $g > 0$). They are identical in the problem above only because δ, n and g are the same in the two countries.

3. Problems and Applications 6: **3.6.** The rate of growth of output Y will fall by one percentage point in the very long run. The rate of growth of output per worker Y/L will be unchanged in the very long run. However, the level of output per worker will be higher in the very long run (the economy will be on a higher growth path with the same rate of growth—2% per year). During the transition to the new very long run path, output per worker will grow a bit faster than 2% per year.



Chapter 9:

4. *Questions for Review* numbers 1,2,4,6: Please see text or lecture notes.

5. Problems and Applications 6:

5.6a,b We can think of a higher level of education in one country as being equivalent to having a better technology. I.e., the same number of workers and machines will be able to produce more output. For example, using Mankiw's notation, if the production function for the economy is $Y = F(K, E \cdot L)$ where E indicates how "efficient" labor is, a country with a higher level of education has a higher level of E. Thus, this country will have a higher level of output per worker, but not a (permanently) higher growth rate than will other countries. The growth rate of output per worker will still be equal in the very long run to the rate of technological progress, which is assumed in this problem not to depend on the level of education in each country.[8]

6. This problem shows that the results of the Solow model change dramatically if we abandon the assumption of diminishing returns to capital. This endogenous growth model is covered in section 9.4 of the text.

The Solow Growth Model is an *exogenous growth model*. Sustainable growth of per capita output and consumption is explained by technological progress which is assumed to be exogenous (i.e., not affected by the actions of people and firms within the economy). Technological progress is assumed to just happen, and is not explained within the model. It is not, for example, dependent on the economy's saving rate.

The model in this problem is a very simple example of an *endogenous growth* model.[9] Without diminishing returns to capital, the cycle of saving, investing, and growing doesn't peter out. Thus, the economy will experience economy growth even in the absence of any technological change. Further, the rate of growth depends on the behavior of people in the economy. If an economy raises its saving rate, it will accumulate capital faster and grow faster even in the very long run.

The per-worker production function is now

$$y = k$$

With no population growth and no technological progress, the equation of motion for the capital stock is

$$\begin{aligned}\Delta k &= s \cdot f(k) - \delta\, k \\ &= s \cdot k - \delta\, k \\ &= (s - \delta)\, k\end{aligned}$$

Since $y = k$,

$$\Delta y = (s - \delta)\, y$$

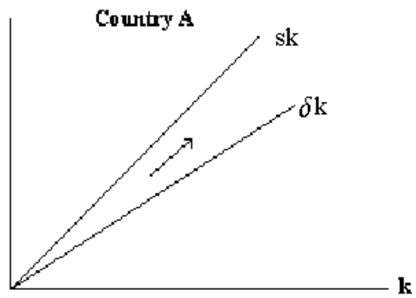
So that

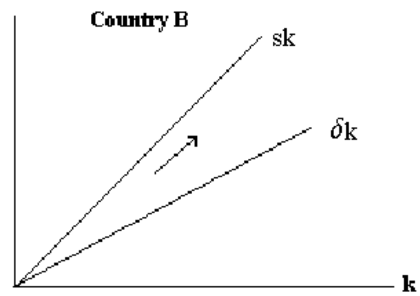
$$\frac{\Delta y}{y} = (s - \delta)$$

This says that output per worker (y), grows at a rate of $(s - \delta)$ per year, all of the time. This means that, not only is there growth in the standard of living in the very long run even in the absence of technological progress, but that the growth rate depends on the saving rate. High saving countries grow at a permanently higher rate than do low saving countries. Thus, standards of living of countries with different saving rates diverge over time.

[8] I.e., the level of the path of E has increased, but it's growth rate g is unchanged.

[9] This simple example is often called the "AK model," as the production function has the form $Y = AK$. In this problem I set $A = 1$.



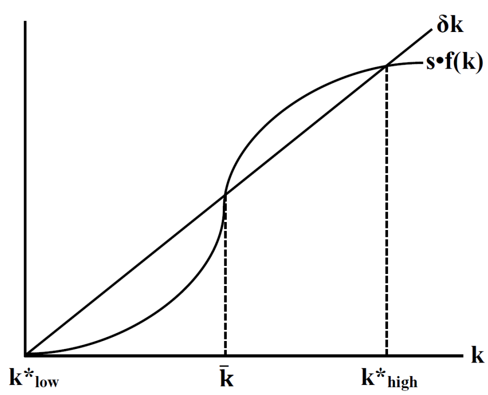


Some economists have argued that this kind of model is better than the Solow model. For example some have argued that, while individual firms probably typically face diminishing marginal products of capital, this need not imply that the economy as a whole does. If, for example, my investment in a technology allows others to learn from my experience or provides increased opportunities for others to subspecialize, these externalities (spillover effects) from investment could make the aggregate production function have non-diminishing MPK.

A second approach to explaining growth endogenously is simply to recognize that technological progress within a country is at least partly the result of investments (in R&D, education, infrastructure, etc.) made by businesses, individuals, and the government within that economy. In that case, if the economy faces diminishing returns to capital, then technological progress is still the key to explaining growth, as it is in the Solow model. However, unlike the Solow model, this argument claims that the country's rate of technological progress depends on its saving rate (saving is needed to finance the investments that generate technological progress). The two sector model in section 9.4 of the text is an example of this kind of endogenous growth model.[10]

If either of these endogenous growth mechanisms is actually important in the real world, this could bode poorly for low saving countries in the very long run.

Now as an aside, extending the analysis in question **6**, there are yet other shapes for the production function that we could consider. For example, suppose that we had the following picture.



In this case, if the economy starts with $k > \bar{k}$, it gravitates to k^*_{high} over time. If the economy starts with $k < \bar{k}$, it gravitates to k^*_{low} over time. Thus, $\bar{k}$ is a kind of development threshold. Economies that start with low capital stocks get trapped at lower standards of living than economies

[10] Parenthetically, if we define the capital stock very broadly (e.g., to include the country's stock of knowledge), then this model looks like an AK model.

that already have high capital stocks. Further, an economy starting just above $\bar{k}$, will grow rapidly toward k^*_{high}, whereas a country starting below $\bar{k}$ will have negative growth.

The key to this variation on the Solow model is that the MPK is very large around $\bar{k}$. In other words, a little extra capital is very productive in this range. This might be explained in terms similar to the discussion in question **6**. It might be that there is a critical level of economic activity at which externalities kick in sufficiently to allow for a rapid development of markets. For example, for high levels of economic activity to be sustainable, there may need to be a variety of different industries which are closely integrated, good secondary education, universities, research parks, etc.

This of course begs the question as to how advanced economies got past $\bar{k}$ while other countries may not have. Clearly this is not merely a matter of technology.

Jeffrey Sachs has argued that some of the poorest countries in the world are stuck in development traps with $k < \bar{k}$ due to factors such as adverse geography or poor political governance, and could be shocked past $\bar{k}$ by very large infusions of well targeted aid from rich countries (a "big push"). Esther Duflo is more cautious about such big claims, as she notes at the beginning of her TedTalk.

Appendix to Chapter 9:

7. By standard growth accounting, we have:

$$\widehat{Y} = \widehat{A} + \alpha \cdot \widehat{K} + (1 - \alpha) \cdot \widehat{L}$$

where A is "total factor productivity" or "TFP" and so $\widehat{A}$ is the rate of TFP growth.

Given the data on the growth rates of Y, K, and L, and the assumption that $\alpha = 1/3$, we can calculate $\widehat{A}$ as a residual (often called the "Solow Residual") from the equation above. We find that TFP growth was somewhat less than one percent (approximately 0.7%). The contributions to output growth from labor growth and the growth of the capital stock were approximately 0.8% and 0.7% respectively.