Economics 504
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Handout on Growth Rates

Discrete Time Analysis:

All macroeconomic data are recorded for discrete periods of time (e.g., quarters, years). Consequently, it is often useful to model economic dynamics in discrete periods of time.

Let's consider annual GDP per capita (gross domestic output for a year divided by the number of people in the population in that year) of a country. Call this Y.

Define the annual growth rate g of Y in any year t as the annual percentage change in Y from the previous year.

$$\begin{aligned} g_t &= \widehat{Y}_t \\ &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} \end{aligned}$$

Defined in this way, growth rates are compounding over time. Starting at time 0, we have

$$\frac{Y_1 - Y_0}{Y_0} = g_1$$

and solving this for Y_1

$$Y_1 = (1 + g_1) \cdot Y_0$$

Similarly,

$$\begin{aligned} Y_2 &= (1 + g_2) \cdot Y_1 \\ &= (1 + g_2) \cdot (1 + g_1) \cdot Y_0 \end{aligned}$$

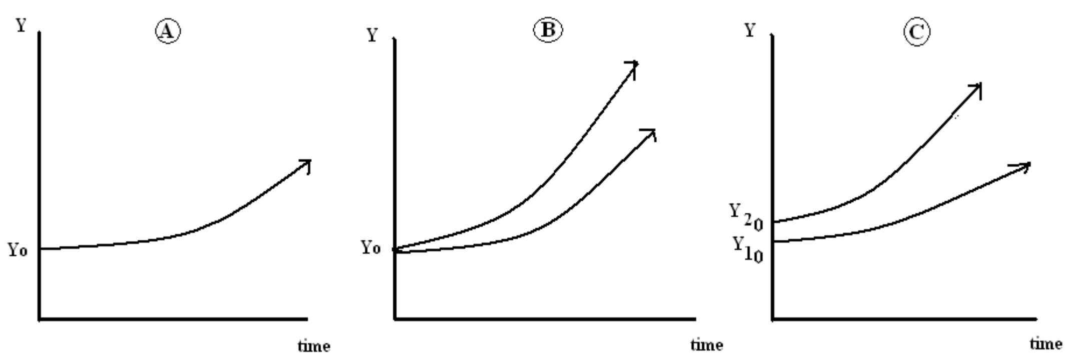
And for any future time t

$$Y_t = (1 + g_1) \cdot (1 + g_2) \cdots (1 + g_t) \cdot Y_0$$

If g_t happened to be constant over the t years following year 0, then

$$Y_t = (1 + g)^t \cdot Y_0$$

and the path of Y over time would look like that in figure A below.



Starting from the same initial level Y_0, a larger growth rate g rotates the curve upward (figure B above). The gap between two economies which start at the same level but grow at different rates grows over time. In other words, due to compounding, small differences in permanent growth rates have large effects in the future.

Consider also two economies (1 and 2) which start at different initial levels Y_0 but grow at the same rate (figure C above). The absolute gap $(Y_{1t} - Y_{2t})$ between the levels of GDP per capita of the two countries grows over time, but the ratio Y_{1t}/Y_{2t} remains constant.

The following approximations for percentage growth rates are useful. For small changes in any two variables x and y:

$$\widehat{x/y} \approx \widehat{x} - \widehat{y}$$
$$\widehat{x \cdot y} \approx \widehat{x} + \widehat{y}$$

For the really interested reader, a proof of first proposition appears in the appendix at the end of this handout.

Application: Post WWII growth in the U.S. and Japan

Here are some measures of per capita real GDP for the US and Japan in 1950 and 1989:

	1950	1989
US	8,611	18,317
Japan	1,563	15,101

What are the annual average growth rates over this period for the US and Japan?

Here is one way to answer this question:

$$Y_{1989} = (1+g)^{39} \cdot Y_{1950}$$

Consequently, g can be calculated

$$(1+g) = \left(\frac{Y_{1989}}{Y_{1950}}\right)^{\frac{1}{39}}$$

Yielding $g = 0.0195$ for the US and $g = 0.0597$ for Japan. The US grew at an average growth rate of about 2% annually over the period while Japan grew at about 6% annually.[1]

Log Growth Rates:

The following method gives a close approximation to the answer above, and will be useful in other contexts. A useful approximation is that for any small number x:

$$\ln(1+x) \approx x$$

Now we can take the natural log of both sides of

$$Y_{1989} = (1+g)^{39} \cdot Y_{1950}$$

to get

$$\ln(Y_{1989}) = 39 \cdot \ln(1+g) + \ln(Y_{1950})$$

which rearranges to

$$\ln(1+g) = \frac{\ln(Y_{1989}) - \ln(Y_{1950})}{39}$$

and using our approximation

$$g \approx \frac{\ln(Y_{1989}) - \ln(Y_{1950})}{39}$$

[1] It is worth noting that these numbers are not the same as the averages of the actual annual growth rates taken year by year — which just goes to show that there is rarely a single correct method for measuring things.

In other words, log growth rates are good approximations for percentage growth rates. Calculating log growth rates for the data above, we get $g \approx 0.0194$ for the U.S. and $g \approx 0.0582$ for Japan. The approximation is close for both, but closer for the U.S. than Japan as the log approximation will be closer, the closer g is to zero. Log growth rates are often used in economic modeling and empirical work. For example, for year to year growth, researchers will often just use the change in the log: $\Delta \ln(Y_t)$.

Log Plots:

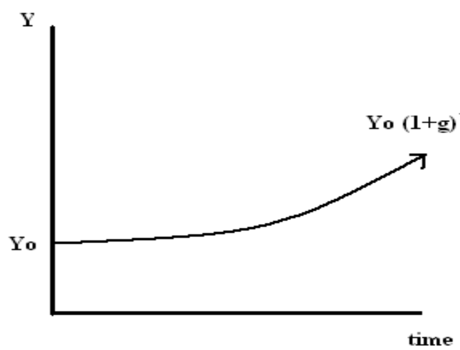
Recall that, with a constant growth rate g and starting from time 0, output in time t is

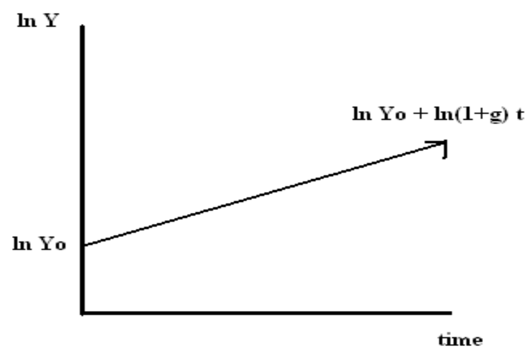
$$Y_t = (1+g)^t \cdot Y_0$$

Taking logs of both sides,

$$\ln Y_t = \ln Y_0 + \ln(1+g) \cdot t$$

we see that log output is *linear* in time. Thus, if the growth rate is constant, a plot of log output against time will yield a straight line. Consequently, plotting log output against time is a quick way to eyeball whether growth rates have changed over time.





Time to Doubling:

How long will it take for standards of living to double? If we measure the standard of living by GDP per capita, for example, then this reduces to the question, in what year t will GDP per capita be twice that of year 0.

To answer this question, we want to solve

$$Y_t = 2Y_0$$

for t.

$$(1+g)^t \cdot Y_0 = 2Y_0$$
$$(1+g)^t = 2$$

Use logs to get t out of the exponent:

$$t \cdot \ln(1+g) = \ln(2)$$
$$t = \frac{\ln(2)}{\ln(1+g)}$$

We can get a good approximation to this by calculating $\ln(2) \approx 0.7$ and using our approximation $\ln(1+g) \approx g$. Thus: $t \approx .7/g$.

Notice that this is smaller than $1/g$ due to compounding. Again, small differences in growth rates have increasingly large effects on future standards of living.

Then in the US for 1950–1989, with $g \approx 0.02$, GDP per capita doubled roughly every 35 years over the period. In Japan with $g \approx 0.06$, GDP per capita doubled roughly every 12 years over the period.

At these kinds of growth rates, successive generations are substantially better off than their predecessors. Notice that if per capita growth falls to 1%, years to doubling rises to about 70 years. A few percentage points in growth rates makes a big difference.

Continuous Time:

For modeling purposes it is sometimes useful (and less clumsy) to work in continuous time. Suppose that we are interested in annual growth patterns, but also want to consider periods of time shorter than a year. In the extreme case we can think of there being a growth rate (measured as an annual rate) at each instant. I.e, the annual rate can be constantly changing, and the actual increase in output over the course of any year depends on all the growth rates during the year (i.e., on average growth during the year).

In this case, rather than defining the growth rate g as the percentage change in GDP from one year to the next, we define it as the *instantaneous* rate of growth of GDP.

$$\begin{aligned} g_t &= \widehat{Y}_t \\ &= \frac{\dot{Y}_t}{Y_t} \\ &= \frac{\frac{dY_t}{dt}}{Y_t} \end{aligned}$$

where $\dot{Y}_t$ is shorthand for the derivative of output with respect to time, $\frac{dY_t}{dt}$.

As in the discrete time case, we need to add up the changes in output over time to calculate future levels of output. However, in continuous time we would do this by integrating over time, which in the case of a constant growth rate g would yield

$$Y_t = Y_0 \cdot e^{g \cdot t}$$

This looks very similar to our formula under discrete time (which was $Y_t = Y_0 \cdot (1+g)^t$), and is close numerically as well.[2]

The analytical convenience of continuous time analysis stems from the fact that the approximations that I discussed above under discrete time are exact equalities under continuous time. The following equalities hold exactly in continuous time:

$$\begin{aligned} \widehat{x/y} &= \widehat{x} - \widehat{y} \\ \widehat{x \cdot y} &= \widehat{x} + \widehat{y} \\ \frac{\ln x_t - \ln x_0}{t} &= g \\ \widehat{x}_t &= \frac{d}{dt} \ln x_t \\ t_{\text{doubling}} &= \frac{\ln 2}{g} \end{aligned}$$

The third and sfourth equation say that log diferrences are exactly equal to the growth rate.

Again, the curious reader can see the appendix for proofs of some of these propositions.

[2] In continuous time, output is slightly greater in the future than it is in discrete time (i.e., $e^{gt} > (1+g)^t$), because growth is compounding continuously rather than annually.

The Solow Residual:

The Solow residual is an empirical measure of total factor productivity (TFP) growth and is often used as a rough measure of the contribution of technological progress to economic growth.

Consider the Cobb Douglas production function with constant returns to scale:

$$Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

The parameter A is total factor productivity TFP. The parameter α is the elasticity of output with respect to capital and also reflects the relative productivities of capital and labor. Empirical estimates of α often put it at around 1/3 (more on that at a later date).

If we take growth rates of each side of this equation and rearrange, we have:[3]

$$\begin{aligned} \widehat{Y} &= \widehat{A} + \alpha \cdot \widehat{K} + (1-\alpha) \cdot \widehat{L} \\ \widehat{A} &= \widehat{Y} - \alpha \cdot \widehat{K} - (1-\alpha) \cdot \widehat{L} \end{aligned}$$

Consider the second equation abovve. If we use an independent estimate of α (like 1/3), then we can take the right hand side of the second equation above (the Solow residual) as an observable measure of the unobserved left hand side (TFP growth).

Appendix

A: Proof of proposition that

$$\widehat{x/y} \approx \widehat{x} - \widehat{y}$$

in discrete time.

Exact method:

$$\begin{aligned} \widehat{x/y} &= \frac{x_1/y_1 - x_0/y_0}{x_0/y_0} \\ &= \frac{x_1}{x_0} \Big/ \frac{y_1}{y_0} - 1 = \frac{1+\widehat{x}}{1+\widehat{y}} - 1 \\ &= \frac{\widehat{x} - \widehat{y}}{1+\widehat{y}} \approx \widehat{x} - \widehat{y} \end{aligned}$$

The approximation makes use of the fact that, for small $\widehat{y}$, $1+\widehat{y}$ is close to 1.

Alternative method: use the fact introduced in the handout that log differences are close approximations to percentage growth rates.

$$\begin{aligned} \widehat{x/y} &\approx \Delta \ln x/y \\ &= \ln x_1/y_1 - \ln x_0/y_0 \\ &= \ln x_1 - \ln x_0 - \ln y_1 + \ln y_0 \approx \widehat{x} - \widehat{y} \end{aligned}$$

B: Proof of proposition that

$$\widehat{x/y} = \widehat{x} - \widehat{y}$$

in continuous time.

The proof makes use of the fact that in continuous time, the time derivative of the log of a variable is the growth rate of that variable. To see this, recall that the derivative of $\ln x$ is $1/x$. Thus, $\frac{d}{dt} \ln x_t = \frac{1}{x_t} \cdot \frac{d}{dt} x_t = \dot{x}_t / x_t = \widehat{x}_t$.

[3] Note that, from what we have learned above, the following equations are exact equalities if we are working in continuous time and close approximations if we are working with discrete time data.

The proof is then:

$$\begin{aligned}
\widehat{x/y} &= \frac{d}{dt}\ln(x/y) \\
&= \frac{d}{dt}(\ln x - \ln y) \\
&= \frac{d}{dt}\ln x - \frac{d}{dt}\ln y \\
&= \widehat{x} - \widehat{y}
\end{aligned}$$