Econ 504
Chris Georges Intro to Return, Risk, and Portfolio Diversification

Diversification:

A core insight in finance is that diversification, i.e., splitting an investment portfolio across a number of different assets or investments, rather than holding "all your eggs in one basket," tends to reduce the riskiness of the portfolio. Intuitively, if each individual investment can idiosyncratically perform well or poorly at a given time, then diversification allows this idiosyncratic variation to average out across the investments in the portfolio, yielding a less variable portfolio return. However, how much diversification reduces this risk depends on the correlation between the returns on the various investments in the portfolio.

Simple Example with Coin Tosses

One Coin Toss: Let's take a simple example. You pay (invest) \$10 to participate in a gamble. There is a single coin toss. If the coin comes up heads, you receive \$12, and if tails, your receive \$9. Thus, you get a (rate of) return on your investment of 20% if heads, and -10% if tails.

Since we can assign probabilities to the two possible outcomes in this example, we can call the **expected return** for the investment the mathematical **expected value** of the return, which is in this case $\frac{1}{2} \cdot 20 + \frac{1}{2} \cdot (-10) = 5\%$. Note that we will never actually receive this expected value in a single gamble, since we either get 20% or -10%.

One way to measure the riskiness of the gamble is to measure how far the actual return is likely to be be from the expected return on average. A standard such measure of **risk** is the **variance** of the return: the average squared deviation of the various actual returns that are possible from the expected return. Since the actual possible deviations for our gamble are 15 and -15, and each occurs with equal probability, the variance of the return in this case is 225. Another common measure is the standard deviation, which is just the square root of the variance, here 15. I.e., a 'standard' deviation of the actual return from the average return for our gamble is 15 percentage points.

Two Coin Tosses: Now let's take our \$10 and divide it into two gambles. We will pay \$5 each to invest in two separate coin tosses. Each one pays \$6 if heads and \$4.50 if tails. I.e., the equally likely possible rates of return are again 20% and -10%. So the expected return to each is 5%, and the expected return to the portfolio of two gambles is also still 5%.

But the risk is now lower on our \$10 investment. The possible outcomes for the two coin tosses are HH, HT, TH, TT, each with probability 1/4. So 1/4 of the time we get TT, and the portfolio returns -10%. 1/2 of the time, there is one head and one tail, and the portfolio returns 5%. And 1/4 of the time we get HH, and the portfolio returns 20%.

So diversification has reduced the spread of the actual outcomes around the average outcome. If you calculate the variance (taking the average of the four coin toss outcomes, or equivalently the weighted average of the three possible returns outcomes using the probabilities 1/4, 1/2, 1/4 as the weights), you will see that the variance of the portfolio return is now 112.5 and so the standard deviation is now approximately 10.6. So the standard variation around the average is now about 10.6 % pts. rather than 15 % pts. Now, half the time, the deviation of the actual return from the expected return is zero, and only half the time is the deviation 15 or -15 % pts.

If we kept dividing this portfolio into more and smaller independent investments, we would see the variance and standard deviation of the portfolio return (i.e., the riskiness of the portfolio) continue to fall towards zero. Indeed, as we will see formally below, these approach zero as the number of independent investment in the portfolio approaches infinity.

Correlation and Diversification: Now in the example above, it is critical that the two coin tosses were independent. If the returns to both investments were the same (e.g., were both determined by the same coin toss), then diversification would have no effect on the riskiness of the portfolio. In general, the more positively correlated are the returns of the various investments in the portfolio, the less effective is a given amount of diversification in reducing the riskiness of the portfolio (i.e., in reducing the variance of the return on the portfolio). This is again shown more formally below. On the other hand, if an investor can find investments that are negatively correlated (when one performs above average, the other tends to perform below average, and vice versa), then it takes less diversification to reduce the riskiness of the portfolio. In the extreme case of perfect negative correlation between two securities, a portfolio of just the two securities can have zero risk (zero variance of the portfolio return).

Models of asset prices, such as CAPM, usually conclude that securities should only command risk premia (extranormal expected returns) for non-diversifiable risk, also called 'systematic' risk or beta. Thus, a security might have a highly uncertain return, but if its covariation with the rest of the market is small, it should have an expected return close to the rates on relatively riskless assets, since much of its risk can be diversified away.

A central cause of the financial crisis of 2007-2008 was that various financial market participants (AIG, Bear Sterns, Countrywide, individual investors) grossly underestimated the systematic risk that they were exposed to. For example, mortgage backed securities were rated and priced on the assumption that housing prices were highly unlikely to fall and mortgage delinquencies rise nationwide, so that pooling mortgages from different regional markets was effectively diversifying away risk.

More on Reducing Risk Through Diversification

We can show that the risk of the return on a financial portfolio can be reduced by diversification among a large number of securities with risky but mutually uncorrelated returns.

Consider holding a portfolio of N securities with weights s_i (s_i is the proportion of the dollar value of the portfolio held in security i) and rates of return r_i (r_i is the rate of return per dollar held in security i).

For a portfolio of dollar value P, the rate of return on the portfolio is

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\begin{aligned}
r_P &= \frac{s_1 P r_1 + s_2 P r_2 + \ldots + s_N P r_N}{P} \\
&= \sum_{i=1}^{N} s_i r_i
\end{aligned}
$$

Each return r_i is a random variable (i.e., individual security returns are risky), and consequently so is the return on the portfolio r_P.

Define the expected (rate of) return on the portfolio as the (mathematical) expected value of this return. Then this expected return is

$$\begin{aligned}\mathbf{E}(r_p) &= \mathbf{E}(\sum_{i=1}^{N} s_i r_i) \\ &= \sum_{i=1}^{N} s_i \mathbf{E}(r_i)\end{aligned}$$

and if we measure the riskiness of the portfolio as the variance of its return, then this riskiness is

$$\begin{aligned}\mathbf{Var}(r_p) &= \mathbf{Var}(\sum_{i=1}^{N} s_i r_i) \\ &= [\sum_{i=1}^{N} s_i^2 \mathbf{Var}(r_i)] + [\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} s_i s_j \mathbf{Cov}(r_i, r_j)]\end{aligned}$$

Further, **if** the returns of the individual securities are mutually uncorrelated, then all the covariances above are zero, so that

$$\mathbf{Var}(r_p) = \sum_{i=1}^{N} s_i^2 \mathbf{Var}(r_i)$$

Now suppose that we hold equal shares of each security, so that $s_i = 1/N \ \ \forall i$, and for simplicity further suppose that the variance of each security's returns is identical: $\mathbf{Var}(r_i) = \sigma^2 \ \ \forall i$. Then for N independent (uncorrelated) securities

$$\begin{aligned}\mathbf{Var}(r_p) &= \sum_{i=1}^{N} (\frac{1}{N})^2 \sigma^2 \\ &= N \cdot (\frac{1}{N})^2 \sigma^2 \\ &= \frac{\sigma^2}{N}\end{aligned}$$

Thus, as we increase the number of independent securities held in the portfolio (i.e., increase the degree of diversification of the portfolio), the portfolio's riskiness becomes smaller and goes to zero in the limit as $N \rightarrow \infty$. Intuitively, as the number of securities increases, the likelihood that the *all* do very well or *all* do very poorly falls, and the likelihood that the realized portfolio return is near its expected value increases.

Note that positive covariances of returns across securities would limit our ability to reduce risk through diversification, and negative covariances would increase our ability to do this with a small number of securities. We can see this above. Without the assumption of zero covariance, the variance of the portfolio depends both on the variances of each security, but also on the covariances of the returns of each pair of securities in the portfolio. The variance of the portfolio return is increased by positive correlation between specific pairs of securities in the portfolio and reduced by negative correlation between specific pairs of securities.

Appendix: A Few Statistical Definitions and Identities

Consider two random variables X and Y defined over m distinct possible events. Event i occurs with probability p_i, in which case X and Y take on values x_i and y_i. Thus the probabilities of the various events occurring are p_1, p_2, ..., p_m, and X and Y take on possible values x_1, x_2, ..., x_m and y_1, y_2, ..., y_m respectively. If we have considered all possible events, then it must be that the sum of the m probabilities is one: $\sum_{i=1}^{m} p_i = 1$.

For example, consider our coin toss, but now with two players, and without the \$10 investment. We flip a coin once. In the event that it lands heads-up, Beth and Bob each get \$1. In the event the coin lands tails-up, Beth gets \$2 and Bob gets nothing. Here there are two possible events (heads and tails), which occur with equal probability $p_1 = p_2 = 0.5$. The payoffs to Beth and Bob are each random variables which we could call X and Y.

We will use the following notation

$$\begin{aligned}
\mathbf{E}(X) &= \text{The Expected Value of } X \\
\mathbf{Var}(X) &= \text{The Variance of } X \\
\mathbf{Cov}(X, Y) &= \text{The Covariance of } X \text{ and } Y \\
\sigma_X &= \text{The Standard Deviation of } X \\
\rho_{XY} &= \text{The Correlation of } X \text{ and } Y \\
\mu_X &= \mathbf{E}(X)
\end{aligned}$$

These quantities are defined as follows.

$$\begin{aligned}
\mathbf{E}(X) &= p_1 x_1 + p_2 x_2 + ... + p_m x_m \\
&= \sum_{i=1}^{m} p_i x_i
\end{aligned}$$

$$\begin{aligned}
\mathbf{Var}(X) &= p_1(x_1 - \mu_X)^2 + p_2(x_2 - \mu_X)^2 + ... + p_m(x_m - \mu_X)^2 \\
&= \sum_{i=1}^{m} p_i(x_i - \mu_X)^2 \\
&= \mathbf{E}[(X - \mu_X)^2] \\
\sigma_X &= \sqrt{\mathbf{Var}(X)} \\
\mathbf{Cov}(X, Y) &= p_1(x_1 - \mu_X)(y_1 - \mu_Y) + p_2(x_2 - \mu_X)(y_2 - \mu_Y) + ... + p_m(x_m - \mu_X)(y_m - \mu_Y) \\
&= \sum_{i=1}^{m} p_i(x_i - \mu_X)(y_i - \mu_Y) \\
&= \mathbf{E}[(X - \mu_x)(Y - \mu_Y)] \\
\rho_{XY} &= \frac{\mathbf{Cov}(X, Y)}{\sigma_X \sigma_Y}
\end{aligned}$$

The expected value of X is an average or mean value of the realizations x which occur with frequencies p. The variance of X is a measure of the average amount of variation of the realizations

x around their mean value. The covariance of X and Y is a measure of the degree to which values of X which are larger than average tend to coincide with values of Y which are larger or smaller than average. For example, a negative covariance indicates that when the realization of X is greater than μ_X, the corresponding realization of Y tends, on average, to be less than μ_Y. The correlation of X and Y is the covariance normalized so that this value falls between -1 and 1. We say that X and Y are perfectly correlated if this correlation is -1 or 1.

Returning to our example of Beth and Bob, above, we have $\mathbf{E}(X) = \$1.50$, $\mathbf{E}(Y) = \$0.50$, $\mathbf{Var}(X) = \mathbf{Var}(Y) = 0.25$, $\sigma_X = \sigma_Y = 0.5$, $\mathbf{Cov}(X, Y) = -0.25$, and $\rho_{XY} = -1$.

Now, let a, b, and c be arbitrary constants. The following identities follow from the definitions above.

$$\begin{aligned}
\mathbf{E}(a) &= a \\
\mathbf{E}(a + X) &= a + \mathbf{E}(X) \\
\mathbf{E}(bX) &= b\mathbf{E}(X) \\
\mathbf{E}(a + bX) &= a + b\mathbf{E}(X) \\
\mathbf{E}(X + Y) &= \mathbf{E}(X) + \mathbf{E}(Y) \\
\mathbf{E}(\sum_{i=1}^{n} X_i) &= \sum_{i=1}^{n} \mathbf{E}(X_i)
\end{aligned}$$

In the last identity, X_1, X_2, ... X_n are n random variables.

$$\begin{aligned}
\mathbf{Var}(a) &= 0 \\
\mathbf{Var}(a + X) &= \mathbf{Var}(X) \\
\mathbf{Var}(bX) &= b^2\mathbf{Var}(X) \\
\mathbf{Var}(X + Y) &= \mathbf{Var}(X) + \mathbf{Var}(Y) + 2\mathbf{Cov}(X, Y) \\
\mathbf{Var}(\sum_{i=1}^{n} X_i) &= \sum_{i=1}^{n} \mathbf{Var}(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{Cov}(X_i, X_j)
\end{aligned}$$

It follows from the last identity that **if** n random variables are mutually uncorrelated, the variance of their sum is equal to the sum of their variances.

$$\begin{aligned}
\mathbf{Cov}(X, Y) &= \mathbf{Cov}(Y, X) \\
\mathbf{Cov}(X, X) &= \mathbf{Var}(X) \\
\mathbf{Cov}(a, X) &= 0 \\
\mathbf{Cov}(a + X, b + Y) &= \mathbf{Cov}(X, Y) \\
\mathbf{Cov}(bX, Y) &= b\mathbf{Cov}(X, Y) \\
\mathbf{Cov}(bX, cY) &= bc\mathbf{Cov}(X, Y) \\
\mathbf{Cov}(X, Y + Z) &= \mathbf{Cov}(X, Y) + \mathbf{Cov}(X, Z)
\end{aligned}$$