The Determining Number of a Cartesian Product

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Abstract

A set S of vertices is a determining set for a graph G if every automorphism of G is uniquely determined by its action on S. The determining number of G, denoted $\mathrm{Det}(G)$, is the size of a smallest determining set. This paper begins by proving that if $G = G_1^{k_1} \square \cdots \square G_m^{k_m}$ is the prime factor decomposition of a connected graph then $\mathrm{Det}(G) = \max\{\mathrm{Det}(G_i^{k_i})\}$. It then provides upper and lower bounds for the determining number of a Cartesian power of a prime connected graph. Further, this paper shows that $\mathrm{Det}(Q_n) = \lceil \log_2 n \rceil + 1$ which matches the lower bound, and that $\mathrm{Det}(K_3^n) = \lceil \log_3(2n+1) \rceil + 1$ which for all n is within one of the upper bound. The paper concludes by proving that if H is prime and connected, $\mathrm{Det}(H^n) = \Theta(\log n)$.

1 Introduction

When studying graph properties that involve the graph automorphism group it is useful to have on hand a small subset of vertices that capture the automorphisms of the graph. Such a set S is called a *determining set* and is defined so that every automorphism of the graph is uniquely determined by its action on S. The *determining number* of the graph is defined to be the size of a smallest determining set. Some graphs have small determining number. For example, an n-cycle has determining number 2 (any two non-antipodal vertices create a determining set) and a graphical regular representation (GRR) [15] has determining number 1 (in fact, any vertex of such a graph is itself a determining set). Some graphs have large determining number. For instance, the n-clique has determining number $n-1$ and the complete bipartite graph $K_{r,s}$ has determining number $r+s-2$.

To see some uses of the determining set, let's look at *distinguishing labelings*. A labeling of the vertices of a graph G with labels from the set

$\{1, \ldots, d\}$ is said to be *d-distinguishing* if no nontrivial automorphism of G preserves the vertex labels [3]. Recent work shows that in many infinite families all large graphs are 2-distinguishable. These include hypercubes Q_n with $n \geq 4$ [5], nontrivial Cartesian powers of a connected graph $G \neq K_2, K_3$ [18], and (with seven small exceptions) 3-connected planar graphs [14]. Since distinguishing labelings involve graph automorphisms, determining sets provide a useful tool for studying them. Albertson and Boutin use determining sets to show that the Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [1] and certain types of geometric cliques [2] belong on the list of 2-distinguishable graphs. In [6] Boutin uses determining sets to answer a question of Wilfried Imrich regarding the size of a smallest label class in a 2-distinguishing labeling of Q_n.

Historically, a number of different sets (some with multiple names) have been defined to identify the vertices of a graph relative to the given set. Let $T \subseteq V(G)$. The set T is called a *determining set* [7] or *fixing set* [12] if every automorphism is uniquely determined by its action on the vertices of T. It is called a *resolving set* [9] or *locating set* [20] if every vertex of G is uniquely identified by its distances from the vertices of T. It has been called a *distinguishing set* [4] if each vertex can be uniquely identified by its set of neighbors that are in T. (Note that a distinguishing set has no relationship to the distinguishing labeling which we defined above.) Finally, T has been called a *locating dominating set* [19] or *beacon set* [10] if none of the sets of neighbors that are in T is empty (that is, T is also a dominating set). Each of these sets is a determining set, but not conversely.

Since a determining set is more general then a resolving set, distinguishing set, or locating dominating set, a determining set can be smaller (sometimes much smaller) than the others. For instance, a smallest determining set for Q_8 has size 3 (Section 4), while a smallest resolving set has size 6 [8], and a locating dominating has size at least 28 (because it must dominate). Further, the size of a determining set can be easier to compute. The size of a smallest resolving set for Q_n is known only for small n and it can take "laborious computations" to find [8]. However, we see in Theorem 3 that for all n, $\mathrm{Det}(Q_n) = \lceil \log_2 n \rceil + 1$.

The paper is organized as follows. Basic definitions and properties of determining sets are given in Section 2, as is a review of the Cartesian product and its automorphisms. The *characteristic matrix*, our primary tool for finding determining sets for Cartesian products, is defined in Section 3. This section goes on to prove that if $G = G_1^{k_1} \Box \cdots \Box G_m^{k_m}$ is the prime factor decomposition of a connected graph then $\mathrm{Det}(G) = \max\{\mathrm{Det}(G_i^{k_i})\}$. A lower bound for the determining number of H^k, a Cartesian power of a prime connected graph, is proved in Section 4. The determining number of $Q_n = K_2^n$ is also found in this section, proving that the lower bound is sharp. Section 5 provides an upper bound for $\mathrm{Det}(H^k)$. A relationship between $\mathrm{Det}(K_m^k) = r$ and the number of $m \times r$ covering matrices is given in Section 6. This relationship is then used to compute $\mathrm{Det}(K_3^k)$, which, for all k, is within one of the upper bound from Section 5. Finally, the bounds from Sections 4 and 5 are used in Section 7 to show that $\mathrm{Det}(H^k)$ has growth that is logarithmic in k.

2 Basics

Determining sets and determining numbers have already been introduced; the formal definitions follow. Note that the determining set is an example of a *base of a permutation group action* [11].

Definition 1. *A subset S of the vertices of a graph G is called a* determining set *if whenever $g, h \in Aut(G)$ so that $g(s) = h(s)$ for all $s \in S$, then $g = h$. The* determining number *of a graph G, denoted $Det(G)$, is the smallest integer r so that G has a determining set of size r.*

Note that a set S is a determining set if and only if each vertex in the graph can be uniquely identified by its own graph properties and its relationship to the vertices in S.

We need to be able to tell when a set is a determining set. This can best be done by looking at stabilizers. Recall that for any subset $S \subseteq V(G)$, $\mathrm{Stab}(S) = \{g \in \mathrm{Aut}(G) \mid g(v) = v, \ \forall v \in S\} = \cap_{v \in S}\mathrm{Stab}(v)$. Let id denote the trivial automorphism.

Proposition 1. [7] *Let S be a subset of the vertices of a graph G. Then S is a determining set for G if and only if $Stab(S) = \{id\}$.*

Proof. If S is a determining set then whenever $g \in \mathrm{Aut}(G)$ fixes each $v \in S$, $g = id$. Thus $\mathrm{Stab}(S) = \{id\}$. Further if $\mathrm{Stab}(S) = \{id\}$ and $g, h \in \mathrm{Aut}(G)$ so that $g(v) = h(v)$ for all $v \in S$ then $h^{-1}g(v) = v$ for all $v \in S \Rightarrow h^{-1}g = id \Rightarrow g = h$. $\Box$

Proposition 2. *Let S be a determining set for G and let $\sigma \in Aut(G)$. Then $\sigma(S)$ is also a determining set for G.*

Proof. Note that $\mathrm{Stab}(\sigma(v)) = \sigma\,\mathrm{Stab}(v)\,\sigma^{-1}$. Then we have $\mathrm{Stab}(\sigma(S)) = \cap_{v\in S}\sigma\,\mathrm{Stab}(v)\,\sigma^{-1} = \sigma\ (\cap_{v\in S}\mathrm{Stab}(v))\ \sigma^{-1} = id$ if S is a determining set. $\square$

Definition 2. *The* Cartesian product *of graphs G and H, denoted by $G \,\square\, H$, has vertex set $V(G) \times V(H)$ with an edge between vertices (x, u) and (y, v) if either x is adjacent to y in G and $u = v$, or u is adjacent to v in H and $x = y$. The* Cartesian power*, H^k, is the Cartesian product of H with itself k times.*

A good reference for Cartesian products is [17]. In particular, automorphisms of Cartesian products are discussed beginning on page 128. Recall that H is *prime* with respect to the Cartesian product if it cannot be written as the Cartesian product of two smaller graphs. Every connected graph can be written uniquely (up to order) as the Cartesian product of prime factors, $G = G_1 \,\square \cdots \square\, G_m$. Further, each automorphism of G is composed of automorphisms of the prime factors and permutations of isomorphic factors. Formally, if $\varphi \in \mathrm{Aut}(G)$ then there is a permutation $\pi \in S_m$ and isomorphisms $\psi_i : G_i \to G_{\pi(i)}$ so that $\varphi(v_1, \ldots, v_m) = (\psi_{\pi^{-1}(1)}(v_{\pi^{-1}(1)}), \ldots, \psi_{\pi^{-1}(m)}(v_{\pi^{-1}(m)}))$.

At various points in the paper we need to know what it means for certain objects to be isomorphic images of each other. Two r-tuples $(v_1, \ldots, v_m)$ and $(u_1, \ldots, u_m)$ with entries from G_i and G_j respectively, are *isomorphic images* of each other if there exists an isomorphism $\varphi : G_i \to G_j$ so that $\varphi(v_i) = u_i$ for all i. A similar definition applies to two columns with entries from G_i and G_j. Two ordered subsets of $V(G)$, $\{V_1, \ldots, V_r\}$, and $\{U_1, \ldots, U_r\}$ are isomorphic images if there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ so that $\varphi(V_i) = U_i$ for all i.

3 Characteristic Matrices

Our primary tool for studying the determining set of a Cartesian product will be the *characteristic matrix*. The definition for characteristic matrix given below is analogous to the one used in coding theory. In general, the structure of Cartesian products makes matrices particularly useful for studying questions involving their automorphisms. See [7], [16], and [13] for further examples of matrix techniques used to find determining sets or distinguishing labelings.

Definition 3. *Let $S = \{V_1, \dots, V_r\}$ be an ordered set of m-tuples. Define the* characteristic matrix $M(S)$ *(or just M) to be the $r \times m$ matrix whose ij^{th} entry is the j^{th} coordinate of V_i.*

In particular given a Cartesian product $G = G_1 \square \cdots \square G_m$, each vertex in $V(G)$ can be written as $V = (v_1, \dots, v_m)$ with $v_i \in G_i$. Thus every (ordered) set of vertices $S \subseteq V(G)$ has a characteristic matrix with respect to that product. The crucial property of a characteristic matrix is that if $G = G_1 \square \cdots \square G_m$ is a prime factor (or relatively prime) decomposition, an automorphism φ of G fixes S if and only if, when applied to the entries of M in the obvious way ($\varphi(m_{ij}) = \psi_{\pi^{-1}(j)}(m_{i\pi^{-1}(j)})$ from the notation for automorphisms of Cartesian products), φ leaves M unchanged. In this situation we can say φ fixes M and therefore S.

Lemma 1. *Let G be a connected graph with prime factor decomposition $G = G_1 \square \cdots \square G_m$. Let $S = \{V_1, \dots, V_r\} \subseteq V(G)$. Then S is a determining set for G if and only if each column of the characteristic matrix M contains a determining set for the appropriate factor of G and no two columns of M are isomorphic images of each other.*

Proof. Suppose that column i of M does not contain a determining set for G_i. Then there is a nontrivial automorphism ψ_i of G_i which fixes all the elements in column i of M. Then the automorphism of G given by $\varphi(v_1, \dots, v_m) = (v_1, \cdots, v_{i-1}, \psi_i(v_i), v_{i+1}, \cdots, v_m)$, (that is, π is trivial and when $j \neq i$, so is ψ_j) is a nontrivial automorphism of G that fixes M and therefore S. Thus S is not a determining set for G.

Suppose that column j is the isomorphic image of column i under $\psi_i : G_i \to G_j$. Let $\pi = (i\ j)$ and let $\psi_j = \psi_i^{-1}$. Then the automorphism of G obtained by composing π with ψ_i and ψ_j provides a nontrivial automorphism of G that fixes M and therefore S. Thus S is not a determining set for G.

Suppose that S is not a determining set. Then there is a nontrivial automorphism given by $\varphi(v_1, \dots, v_m) = (\psi_{\pi^{-1}(1)}(v_{\pi^{-1}(1)}), \dots, \psi_{\pi^{-1}(m)}(v_{\pi^{-1}(m)}))$ that fixes S. Suppose that π is nontrivial. Then there exists i so that $\pi(i) = j \neq i$. Because φ fixes S, it fixes M, and thus columns i and j are isomorphic images of each other. Suppose that π is trivial. Then since φ is not trivial there is some nontrivial $\psi_i : G_i \to G_i$. Since φ fixes S, it fixes M and therefore column i of M. Then the elements in column i of M are not a determining set for G_i. $\square$

Our first goal is to study the determining number of G in terms of a particular form of the prime factor decomposition for G. Since the Cartesian product is associative and commutative, a connected graph can be written uniquely (up to order) as $G = G_1^{k_1} \square \cdots \square G_m^{k_m}$ where the G_i are connected, prime, and distinct, and $G_i^{k_i}$ is the Cartesian product of k_i copies of G_i. Recall that two graphs are *relatively prime* with respect to the Cartesian product if their prime factor decompositions share no common factor. Then the prime factor decomposition $G = G_1^{k_1} \square \cdots \square G_m^{k_m}$ is a maximal decomposition of G into relatively prime factors.

Though Lemma 1 is the natural tool for us to use throughout the paper, for the proof of the first theorem we need a corollary. Note that if $G = G_1 \square \cdots \square G_m$ is a decomposition into relatively prime factors, then the elements of $\mathrm{Aut}(G)$ are composed of automorphisms of the factors. This leads directly to the necessary corollary from which the first theorem immediately follows.

Corollary 2.1. *Let G be a connected graph and $G = G_1 \square \cdots \square G_m$ a decomposition in which every pair of factors is relatively prime with respect to the Cartesian product. Let $S = \{V_1, \ldots, V_r\}$ be a subset of $V(G)$. Then S is a determining set for G if and only if each column of the characteristic matrix M contains a determining set for the appropriate factor of G.*

Theorem 1. *Let $G = G_1^{k_1} \square \cdots \square G_m^{k_m}$ be the prime factor decomposition for a connected graph G. Then $Det(G) = \max\{Det(G_i^{k_i})\}$.*

Proof. By Corollary 2.1 since each column of the characteristic matrix M for a determining set for G must contain a determining set for the appropriate factor, the column length of M must be at least $\max\{\mathrm{Det}(G_i^{k_i})\}$. Further a $\max\{\mathrm{Det}(G_i^{k_i})\} \times m$ characteristic matrix, and its resulting determining set, is easy to construct from determining sets of the given factors. $\square$

Thus to bound the determining number of an arbitrary graph, it is sufficient to bound $\mathrm{Det}(H^k)$, the determining number of a Cartesian power of a prime connected graph.

4 A Sharp Lower Bound

Theorem 2. *Let H be a connected graph that is prime with respect to the Cartesian product. Then $Det(H^k) \geq \max\{Det(H), \lceil \frac{\log k + \log |Aut(H)|}{\log |V(H)|} \rceil\}$. Further this bound is sharp.*

Proof. Suppose that H^k has determining set $S = \{V_1, \ldots, V_r\} \subset V(H^k)$.

By Lemma 1 each column of the characteristic matrix M contains a determining set for H and none is an isomorphic copy of another. How many such columns can there be? There are $|V(H)|^r$ different columns of length r that can be constructed from elements of H. How many different isomorphic images of a given column C are there? We say an automorphism of H is in the stabilizer of column C, denoted $\mathrm{Stab}(C)$, if it is in the stabilizer of each entry of C. Then there are $\frac{|\mathrm{Aut}(H)|}{|\mathrm{Stab}(C)|}$ different isomorphic copies of C. But since each column contains a determining set for H, and by Proposition 1 a determining set has trivial stabilizer, $\mathrm{Stab}(C)$ is trivial. Thus each column has precisely $|\mathrm{Aut}(H)|$ different isomorphic images. Then there are at most $\frac{|V(H)|^r}{|\mathrm{Aut}(H)|}$ columns in M.

Thus $k \leq \frac{|V(H)|^r}{|\mathrm{Aut}(H)|} \Rightarrow r \geq \lceil \frac{\log k + \log |\mathrm{Aut}(H)|}{\log |V(H)|} \rceil$.

Each column of M must also contain a determining set for H. This implies that $r \geq \mathrm{Det}(H)$.

Thus $r \geq \max\{\mathrm{Det}(H), \lceil \frac{\log k + \log |\mathrm{Aut}(H)|}{\log |V(H)|} \rceil\}$.

By Theorem 3 below, $Q_n = K_2^n$ has determining number $\lceil \frac{\log n + \log 2}{\log 2} \rceil = \lceil \log_2 n \rceil + 1$. Thus our lower bound is sharp. $\square$

Recall that the *n-dimensional hypercube*, Q_n, is the n^{th} Cartesian power of K_2. That is, $Q_n = K_2^n$. A standard way to represent Q_n is to identify its vertices with the binary strings of length n. (We think of each factor of K_2 as having a vertex labeled 1 and a vertex labeled 0.) Two vertices in this representation are adjacent in Q_n if their associated binary strings differ in one coordinate position. (That is, adjacent vertices agree in all factors but one and are adjacent in the one.)

Theorem 3. $Det(Q_n) = \lceil \log_2 n \rceil + 1$.

Proof. For the moment, let $n = 2^r$. We will find $r + 1$ vertices which form a determining set for Q_{2^r}. Consider the coordinate positions of the n-bit binary string to be the 1^{st} position, the 2^{nd} position, etc., working from the left.

Let V_0 be the string of all zeros. For $1 \leq i \leq r$, let V_i be the 2^r-bit binary string that alternates 2^{i-1} consecutive ones with 2^{i-1} consecutive zeros. For example, in Q_8 we have $V_0 = 00000000, V_1 = 10101010, V_2 = 11001100$, and $V_3 = 11110000$. More formally, let V_i be the n-bit binary string with ones in positions $t2^i + s$ where $1 \leq s \leq 2^{i-1}$ and $0 \leq t \leq 2^{r-i} - 1$ and zeros in all other positions.

Let $S = \{V_0, \dots, V_r\}$ and let M be the characteristic matrix for S. Since $\mathrm{Det}(K_2) = 1$, each column of M contains a determining set for K_2. Since the first entry of each column is 0, two columns are isomorphic images only if they are equal. Thus by Lemma 1, S is a determining set if and only if no pair of columns is equal.

Suppose that there is a pair i, j so that columns i and j are equal. Then the i^{th} and j^{th} entries in row $r+1$ (the i^{th} and j^{th} coordinates of V_r) are equal. But since the first 2^{r-1} entries of row $r+1$ are 1's and the second 2^{r-1} are 0's, this means that i and j are either both in the first 2^{r-1} entries or they are both in the second 2^{r-1}. In particular then $|i-j| < 2^{r-1}$. But the i^{th} and j^{th} entries in row r (the i^{th} and j^{th} coordinates of V_{r-1}) are also equal. This means that i and j are either both in the first 2^{r-2} positions of the half in which they live or they are both in the second 2^{r-2}. In particular $|i-j| < 2^{r-2}$. Continuing in this manner we can conclude that $|i-j| < 2^{r-1}, \dots, 2^1, 2^0$. Thus $i = j$ and S is a determining set for Q_n.

For $2^{r-1} < n < 2^r$ simply discard any $2^r - n$ columns of M to create an $(r+1) \times n$ matrix with the required properties. This matrix is a characteristic matrix for a determining set of size $r + 1 = \lceil \log_2 n \rceil + 1$ for Q_n. $\square$

5 A Good Upper Bound

Given a connected graph H that is prime with respect to the Cartesian product, if we know a determining set for H, we can quickly and easily create a determining set for H^k in the following way. Let $\{v_1, \cdots, v_r\}$ be a determining set for H of size $r = \max\{\mathrm{Det}(H), 2\}$. Let $S = \{V_1, \cdots, V_{k(r-1)+1}\}$ where $V_1, \dots, V_r$ have first coordinates $v_1, \dots, v_r$ respectively and all other coordinates v_1; $V_{r+1}, \dots, V_{2r-1}$ have second coordinates $v_2, \dots, v_r$ respectively and all other coordinates v_1. In general, for $1 \leq j \leq k$ let $V_{(j-1)(r-1)+2}$ through

$V_{j(r-1)+1}$ have j^{th} coordinates $v_2, \ldots, v_r$ respectively and all other coordinates v_1.

Let M be the characteristic matrix associated with S. Note that by construction each column of M contains $\{v_1, \ldots, v_r\}$, a determining set for H. Further note that if $i \neq j$, column j has v_1 in position 1 and v_2 in position $(j-1)(r-1)+2$ while column i has v_1 in both position 1 and position $(j-1)(r-1)+2$. Since no automorphism can both fix v_1 and transpose it with v_2, columns i and j are not isomorphic images of each another. Thus by Lemma 1, S is a determining set for H^k.

Example 1. Consider K_3^2. The determining number of K_3 is 2, but all determining sets of size 2 in K_3 are isomorphic images of each other. Thus if we tried to find a determining set of size 2 for K_3^2 we would fail; the columns of the resulting characteristic matrix would necessarily be isomorphic images of each other. However, the work above shows that $S = \{(1,1),(2,1),(1,2)\}$ is a determining set for K_3^2. This is a minimal size determining set for this graph.

Though the above is an easy way to find a determining set for H^k, the upper bound this yields, $\text{Det}(H^k) \leq 1 + k \cdot \max\{\text{Det}(H) - 1, 1\}$, is not sharp. However, Theorem 4 below provides an upper bound that is within one of the actual value for at least one infinite family of graphs, K_3^k (see Section 6). This bound is written using s, the number of nonisomorphic ordered determining sets of minimal size for H. However, if we don't know s, we can get a slightly weaker bound by letting $s = 1$.

Theorem 4. *Let H be a connected graph that is prime with respect to the Cartesian product. Suppose that $Det(H) = r$ and that there are s non-isomorphic ordered determining sets of size r for H. Then $Det(H^k) \leq r + \left\lceil \frac{\log k - \log s}{\log |V(H)|} \right\rceil$.*

Proof. First suppose that $k = s|V(H)|^R$ for some nonnegative integer R and construct an $(r+R) \times k$ characteristic matrix M in the following way. Let $S_1, \ldots, S_s$ be a maximal collection of nonisomorphic ordered determining sets of size r for H. For each i, create $|V(H)|^R$ distinct columns of M whose first r entries are the (ordered) entries of S_i.

By construction each column of M contains a determining set for H. Two columns with matching entries in the first r positions cannot be isomorphic images because any automorphism that fixes the first r entries fixes a determining set for H and is therefore trivial on the associated factors. Thus the remaining R entries in each of these two columns would also be fixed and therefore the columns would be equal. Two columns with differences in their first r entries cannot be isomorphic images because their first r entries are nonisomorphic ordered determining sets. Thus M is a characteristic matrix for a determining set for H^k of size $r+R$.

For $k < s|V(H)|^R$ simply discard any $s|V(H)|^R - k$ columns of M to create a $(r+R) \times k$ matrix with the required properties. The resulting matrix is a characteristic matrix for a determining set of size $r+R$ for H^k.

Thus $\mathrm{Det}(H^k) \leq r + R = r + \left\lceil \frac{\log k - \log s}{\log |V(H)|} \right\rceil$. □

Example 2. Consider C_5 and label its vertices $1, 2, 3, 4, 5$ in the obvious way. We know that $\mathrm{Det}(C_5) = 2$. It is easy to see that a maximal collection of nonisomorphic ordered determining sets of C_5 of size two contains two such sets. Thus if $1 \leq k \leq 2$ we can create a determining set of size two for C_5^k and this is the best we can do. If $3 \leq k \leq 10$ we can create a determining set of size three. However, if $11 \leq k \leq 14$ we can still find a determining set of size 3 for C_5^k using the characteristic matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 \end{bmatrix},$$

and dropping columns as necessary to obtain the necessary size.

Notice that there are four more nonisomorphic columns containing determining sets than promised by the proof of Theorem 4.

Example 3. Consider the Petersen graph P. We know that $\mathrm{Det}(P) = 3$ [7]. We can show (by exhaustion) that a maximal collection of nonisomorphic ordered determining sets of size three contains four such sets. Thus by Theorem 4 we have $\mathrm{Det}(P^k) \leq 3 + \lceil \frac{\log k - \log 4}{\log 10} \rceil = 3 + \lceil \log_{10} \frac{k}{4} \rceil$. Then for $5 \leq k \leq 40$, $\mathrm{Det}(P^k) \leq 4$, and for $41 \leq k \leq 400$, $\mathrm{Det}(P^k) \leq 5$. However, in a manner similar to the previous example we can show that there are many k bigger than 41 for which $\mathrm{Det}(P^k) = 4$.

6 Cartesian Powers of Cliques

The following is a useful tool for finding determining sets of Cartesian powers of cliques.

Definition 4. *A* covering matrix *is a* $\{0,1\}$*-matrix that has distinct rows and a single* 0 *in each column. Two such matrices are equivalent if there is a row permutation taking one to the other.*

An $m \times r$ covering matrix is so named because of its correspondence with m-block coverings of $[r]$.

Theorem 5. *The maximum* k *for which* $Det(K_m^k) = r$ *is equal to the number of nonequivalent* $m \times r$ *covering matrices.*

Proof. Using Lemma 1 we see that K_m^k has a determining set of size r if and only if there are there are (at least) k nonisomorphic ordered r-tuples with entries from $V(K_m)$ (nonisomorphic columns for the characteristic matrix) each of which contains at least $m-1$ different elements (a determining set for K_m). Below is a natural bijection between the desired r-tuples and the set of nonequivalent $m \times r$ covering matrices.

Let A be an $m \times r$ covering matrix. Let $\vec{v}$ be the r-tuple of elements of $[m]$ which has i^{th} coordinate equal to the position of the 0 entry in the i^{th} column of A. Note that isomorphic images of $\vec{v}$ correspond to covering matrices equivalent to A. Further note that if there is $j \in [r]$ so that j is not a coordinate of $\vec{v}$, then row j of A contains all 1s. Since no two rows of A are equal, no two elements of $[m]$ are missing from $\vec{v}$, and thus $\vec{v}$ contains a determining set for K_m. Thus the set of of nonisomorphic columns of length r with entries from $[m]$ in which each column contains at least $m-1$ distinct elements corresponds to the set of nonequivalent $m \times r$ covering matrices. The size of this set is the maximum number of columns in a characteristic matrix for a determining set for a Cartesian power k of K_m. Thus it is the maximum value of k. $\square$

Corollary 2.2. $Det(K_3^k) = \lceil \log_3(2k+1) \rceil + 1$.

Proof. By Theorem 5 it is sufficient to find the number of nonequivalent $3 \times r$ covering matrices. There are 3^r ways to place a single 0 in each of r columns. However, this produces three matrices each of which contains a zero row and therefore two equal rows of all 1s. Since there are 3! ways to permute the rows of a matrix, each matrix has 3! covering matrices to which it is equivalent. Thus there are precisely $\frac{3^r-3}{6} = \frac{3^{r-1}-1}{2}$ nonequivalent $3 \times r$ covering matrices. Thus if $\mathrm{Det}(K_3^k) = r$ then $\frac{3^{r-2}-1}{2} < k \leq \frac{3^{r-1}-1}{2}$ and the result follows. $\square$

Comparing $\mathrm{Det}(K_3^k) = 1 + \lceil \log_3(2k+1) \rceil$ to the upper bound of $2 + \lceil \log_3 k \rceil$ from Theorem 4 shows that for each k the upper bound is within one of the actual value.

7 Growth

Theorem 6. *If H is a connected graph that is prime with respect to the Cartesian product, then $Det(H^k) = \Theta(\log k)$.*

Proof. Let $\mathrm{Det}(H) = r$ and suppose that there are s nonisomorphic ordered determining sets of size r for H. Combining the bounds from Theorem 2 and Theorem 4 we get that $\max\{\mathrm{Det}(H), \lceil \frac{\log k + \log|\mathrm{Aut}(H)|}{\log|V(H)|} \rceil\} \leq \mathrm{Det}(H^k) \leq r + \left\lceil \frac{\log k - \log s}{\log|V|(H)|} \right\rceil$. Since H is fixed, $r, s, |V(H)|, \mathrm{Det}(H)$, and $|\mathrm{Aut}(H)|$ are constant, and therefore both upper and lower bounds grow logarithmically with k. $\square$

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