

Distinguishing Geometric Graphs

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Abstract

We begin the study of distinguishing geometric graphs. Let \overline{G} be a geometric graph. An automorphism of the underlying graph that preserves both crossings and noncrossings is called a *geometric automorphism*. A labelling, $f : V(\overline{G}) \rightarrow \{1, 2, \dots, r\}$, is said to be *r-distinguishing* if no nontrivial geometric automorphism preserves the labels. The *distinguishing number* of \overline{G} is the minimum r such that \overline{G} has an r -distinguishing labelling. We show that when \overline{K}_n is not the nonconvex \overline{K}_4 , it can be 3-distinguished. Furthermore when $n \geq 6$, there is a \overline{K}_n that can be 1-distinguished. For $n \geq 4$, $\overline{K}_{2,n}$ can realize any distinguishing number between 1 and n inclusive. Finally we show that every $\overline{K}_{3,3}$ can be 2-distinguished. We also offer several open questions.

1 Introduction

Given an abstract graph G , let \overline{G} denote a geometric graph that realizes G . In particular, $V(\overline{G}) = \{x_1, x_2, \dots, x_n\}$ is a set of n points in general position in \mathbb{R}^2 and the edge $x_i x_j \in E(\overline{G})$ is the straight line segment joining the corresponding pair of points. Two edges, say uv and xy are said to *cross* if the interiors of the line segments from u to v and x to y have nonempty intersection. See [5] for a concise survey of geometric graphs and [6] for a selection of papers on geometric graph theory.

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To consider distinguishing geometric graphs it is necessary to have a concept of when two geometric graphs are equivalent. Two (abstract) graphs are isomorphic if there is an identification of their vertices that preserves both adjacency and nonadjacency. That is, graphs G and H are *isomorphic* if there exists a one-to-one, onto map $\phi : V(G) \rightarrow V(H)$ such that $\phi(u)\phi(v) \in E(H)$ if and only if $uv \in E(G)$. Similarly we say that two geometric graphs are geometrically isomorphic if there is a graph isomorphism that preserves both crossings and noncrossings. Formally, geometric graphs \overline{G} and \overline{H} , are *geometrically isomorphic* if there exists ϕ , a graph isomorphism from G to H , such that $\phi(u)\phi(v)$ and $\phi(x)\phi(y)$ cross in \overline{H} if and only if uv and xy cross in \overline{G} . A geometric isomorphism from \overline{G} to itself is called a *geometric automorphism*. Let $\text{Aut}(\overline{G})$ denote the group of geometric automorphisms of \overline{G} . It is immediate that if \overline{G} is a geometric realization of G , then $\text{Aut}(\overline{G})$ is a subgroup of $\text{Aut}(G)$. We say that subgraph \overline{G}_2 is a *geometric image* of subgraph \overline{G}_1 if there exists $\sigma \in \text{Aut}(\overline{G})$ such that $\sigma(\overline{G}_1) = \overline{G}_2$.

Inspired by Frank Rubin's recreational puzzle [7], Albertson and Collins introduced distinguishing in [1]. Given an abstract graph G , a labelling $f : V(G) \rightarrow \{1, 2, \dots, r\}$ is said to be r -distinguishing if the only element in $\text{Aut}(G)$ that preserves the labels is the identity. Formally, f is *r -distinguishing* if $\phi \in \text{Aut}(G)$ and $f(\phi(x)) = f(x)$ for all $x \in V(G)$ implies that $\phi = id$. The *distinguishing number* of G , denoted by $D(G)$, is the minimum r such that G has an r -distinguishing labelling. It is easy to see that a labelling is distinguishing if and only if every vertex in the graph can be uniquely identified by its vertex orbit and its label. In practice, a labelling is distinguishing if each vertex in the graph can be uniquely identified by a combination of its graph theoretic properties and its label.

Rubin's puzzle is (in modern terminology) to determine $D(C_n)$, the distinguishing number of the n -cycle. One attraction of the puzzle is its contrast. While $D(C_3) = D(C_4) = D(C_5) = 3$, when $n \geq 6$, $D(C_n) = 2$. To see this, consecutively label the vertices of C_n ($n \geq 6$) with $1, 1, 2, 1, 2, \dots, 2$. The vertex labelled 1 with no neighbors labelled 1 is distinguished. Similarly the vertex labeled 2 with no neighbors labelled 2 is distinguished. We can call these the isolated vertices labelled 1 and 2. The two nonisolated vertices labelled 1 can be distinguished by their distance from the isolated 1 (since $n \geq 6$). Similarly the nonisolated 2's can be distinguished from each other by their distance from the isolated 1 (since $n \geq 6$). We will use both this result and this type of argument later in the paper.

It is immediate that $D(K_n) = n$ and when $q > p$, $D(K_{p,q}) = q$. It is straightforward to see that $D(K_{n,n}) = n + 1$. Results in [1] also include: if $\text{Aut}(G)$ is abelian but not trivial, then $D(G) = 2$; if $\text{Aut}(G)$ is dihedral, then $D(G) \leq 3$; if $\text{Aut}(G) \cong S_4$, then $D(G)$ is either 2 or 4 (but not 3); and for any group Γ , there exists a graph G such that $\text{Aut}(G) \cong \Gamma$ and $D(G) = 2$. Tymoczko has considered distinguishing group actions and has shown that if a group Γ acts on a set X , and $|\Gamma| \leq k!$, then X can be k -distinguished [9]. It is immediate that the analogous result holds for graphs. Other results on distinguishing are contained in [2, 3, 4, 8].

We now extend the notion of distinguishing to geometric graphs. Given a geometric graph \overline{G} , a labelling $f : V(\overline{G}) \rightarrow \{1, 2, \dots, r\}$ is said to be r -distinguishing if the only element in $\text{Aut}(\overline{G})$ that preserves the labels is the identity. Formally f is r -distinguishing if $\phi \in \text{Aut}(\overline{G})$ and $f(\phi(x)) = f(x)$ for all $x \in V(\overline{G})$ implies that $\phi = \text{id}$. The *distinguishing number* of the geometric graph \overline{G} , denoted by $D(\overline{G})$, is the minimum r such that \overline{G} has an r -distinguishing labelling. It is immediate that if \overline{G} is a geometric realization of G , then $D(\overline{G}) \leq D(G)$. If $D(\overline{G}) = 1$, we know that $\text{Aut}(\overline{G}) = \{\text{id}\}$, and we say that \overline{G} is *rigid*.

It is natural to look for geometric realizations that require a large number of labels to distinguish. In some sense such realizations have a lot of symmetry. Consequently, we define $D_g(G)$, the *geometric distinguishing number* of the abstract graph G , by $D_g(G) = \max \{D(\overline{G}) : \overline{G} \text{ is a geometric realization of } G\}$.

Example 1. Figure 1 depicts distinguishing labellings of the two distinct geometric realizations of C_4 . In the first \overline{C}_4 , the two vertices labelled 1 are distinguished since one is adjacent to a vertex labelled 2 while the other is not. One can check that no labelling with two labels destroys all nonidentity automorphisms. In the second \overline{C}_4 , the vertices labelled 2 are distinguished by their adjacencies to the vertex labelled 1. One is joined by an uncrossed edge, one is joined by a crossed edge, and one is not adjacent at all. However, if all vertices were labelled 1, there would still be a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and the geometric graph would not be distinguished. Thus we have $D(\overline{C}_4) = 3$ if there are no crossings, and $D(\overline{C}_4) = 2$ if there is a crossing.

This paper begins the study of distinguishing geometric graphs. Section 2 presents tools that we use in our distinguishing results. In Section 3 we show that with the exception of the nonconvex \overline{K}_4 every complete geometric graph

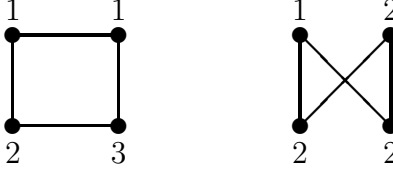


Figure 1: The two \overline{C}_4 's with distinguishing labellings

can be 3-distinguished. We show that when $n \geq 6$ there exists a rigid \overline{K}_n , and that when $n \leq 5$ there is no rigid \overline{K}_n . In Section 4 we consider complete bipartite geometric graphs. We show that when $n \geq 4$, there is a $\overline{K}_{2,n}$ realizing each distinguishing number between 1 and n inclusive. Furthermore, we show that $D_g(K_{3,3}) = 2$. Section 5 closes with open questions.

2 Tools

This section collects some technical definitions and lemmas that are used, sometimes implicitly, throughout this paper.

Given a geometric graph \overline{G} , let $cr(e) = cr_{\overline{G}}(e)$ denote the number of edges that cross the edge e . Let $E_0 = E_0(\overline{G})$ (resp. $E_+ = E_+(\overline{G})$) denote the set of edges in \overline{G} that have $cr(e) = 0$ (resp. $cr(e) > 0$). If $|E_+| = 0$ we say that \overline{G} is a *plane* drawing of G . Let H_0 (resp. H_+) denote the spanning subgraph of \overline{G} whose edge set is E_0 (resp. E_+). We usually think of H_0 and H_+ as abstract graphs though we will occasionally use \overline{H}_0 or \overline{H}_+ to specify a particular geometric realization. Since H_0 and H_+ are each spanning subgraphs and are invariant under every geometric automorphism of \overline{G} , we immediately get the following.

Lemma 1. $D(\overline{G}) \leq \min\{D(H_0), D(H_+)\}$.

Given \overline{G} , if e_1, e_2, \dots, e_s denote the edges incident with the vertex x , then $cr(e_1), cr(e_2), \dots, cr(e_s)$ is called the *crossing sequence* of (the edges incident with) x . Note that the order in the crossing sequence does not matter; we will commonly list them in order of increasing magnitude. The crossing sequence of each vertex in the second \overline{C}_4 shown in Figure 1 is 0, 1. The term *crossing pattern* will be used to speak generally about the crossing sequences of a set of vertices, usually those of a particular subgraph. By definition

crossing sequences and crossing patterns must be preserved by geometric automorphisms.

It will be useful to know when a vertex can be connected to all three vertices of a 3-cycle without crossing any edge of the cycle itself. Given $\triangle ABC$ we define the *outer-angle at A* (with respect to $\triangle ABC$), denoted by A^\angle , to be the angle opposite to $\angle BAC$ in the intersection of lines BA and CA . We show all the outer-angles of $\triangle ABC$ in Figure 2.

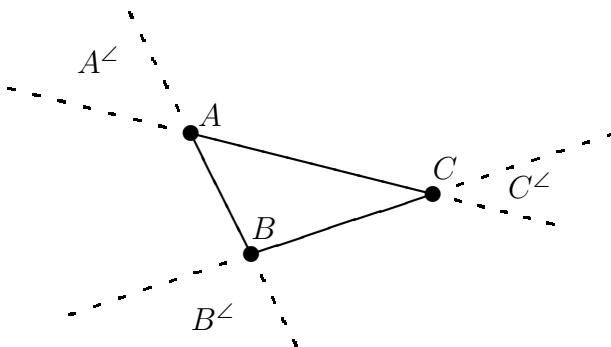


Figure 2: The outer-angles of $\triangle ABC$

It is straightforward to check the following result.

Lemma 2. Suppose A, B , and C are the three vertices of a 3-cycle. A fourth vertex D creates no crossing when connected to A, B and C if and only if D lives in the interior of $\triangle ABC$ or in A^\angle, B^\angle , or C^\angle .

3 Distinguishing \overline{K}_n

We have already observed that $D(\overline{K}_n) \leq D_g(K_n) \leq D(K_n)$. When $n \leq 3$, these numbers are all equal. There are two geometric realizations of \overline{K}_4 . If it is not convex, then \overline{K}_4 contains no crossing edges and $D(\overline{K}_4) = D(K_4) = 4$. In contrast, if \overline{K}_4 is convex, we can label the vertices on the convex hull $1, 1, 2, 3$ in clockwise order. Then the two vertices that are labelled 1 are distinguished by the vertices to which they are adjacent by uncrossed edges. Thus in the convex case, $D(\overline{K}_4) = 3$. When $n \geq 5$ we will see that $D(\overline{K}_n) \leq$

3. The proof will be by induction. Here is an outline. If \overline{K}_n is not vertex transitive and if no geometric vertex orbit is a nonconvex \overline{K}_4 , we could use an inductive hypothesis to 3-distinguish \overline{K}_n one orbit at a time. Thus the two things we need to prove are 1) if \overline{K}_n is vertex transitive but not a nonconvex \overline{K}_4 then $D(\overline{K}_n) \leq 3$; and 2) a vertex orbit that is a proper subset of $V(\overline{K}_n)$ cannot induce a nonconvex \overline{K}_4 . These facts are set out in Lemmas 3 and 4. We follow these with the formal statement and proof of the theorem.

Lemma 3. If \overline{K}_n is vertex transitive, then either

- (a) $n = 4$ and \overline{K}_4 is nonconvex;
- (b) $n = 6$ and \overline{K}_6 contains two disjoint, nested, uncrossed 3-cycles; or
- (c) \overline{K}_n is convex

In the second and third cases $D(\overline{K}_n) \leq 3$.

Proof. Suppose that \overline{K}_n is vertex transitive and nonconvex.

Let C denote the convex hull of \overline{K}_n . Then C is an r -cycle of uncrossed edges. So are all the geometric images of C . The fact that C and its geometric images are cycles of uncrossed edges is crucial to the arguments below.

By assumption \overline{K}_n is not convex so $r < n$. Since \overline{K}_n is vertex transitive, there exists a distinct geometric image of C , say C_1 . Let \overline{K}_C (resp. \overline{K}_{C_1}) be the geometric complete graph induced by the vertices of C (resp. C_1). Note that \overline{K}_{C_1} is a geometric image of \overline{K}_C .

Suppose for the moment that $r > 3$.

Suppose C_1 is nonconvex. Then there is a pair of vertices that are consecutive on the convex hull of C_1 whose preimages in C were not consecutive on its convex hull. This pair of vertices is connected by an uncrossed edge in \overline{K}_{C_1} but by a crossed edge in \overline{K}_C . This is impossible since these edges are geometric images of each other. Thus C_1 cannot be nonconvex.

Suppose C_1 is convex. If a vertex of G is interior to C_1 then an edge from it to a vertex of $V(C) - V(C_1)$ necessarily crosses an edge of the cycle C_1 . This is a contradiction. Thus, no vertex of G is interior to C_1 . Also, no edge can cross to the interior of C_1 without crossing an edge of C_1 itself. Thus edges of \overline{K}_{C_1} are only crossed by edges of \overline{K}_C . Then C_1 must live within one of the polygonal regions determined by the edges of \overline{K}_C . Since $r > 3$ there is at least one vertex of C , say x , that is not on the boundary of this polygonal region. When x is connected to the vertices of C_1 it crosses into the polygonal region in which C_1 lives. But this means that there are edges

connecting C to C_1 that cross edges of \overline{K}_C . Thus the edges of \overline{K}_C are crossed by edges other than edges of \overline{K}_C . This is again a contradiction since \overline{K}_C and \overline{K}_{C_1} must be geometric images of each other. Thus C_1 is also not convex.

We now know that if \overline{K}_n is vertex transitive, but not convex, its convex hull is a 3-cycle.

First, suppose that C_1 shares only a single vertex with C . By transitivity the other vertices of C_1 must also share a single vertex with a geometric copy of C . Thus there is a geometric copy, say C_2 , sharing a single vertex with C_1 . Note that C_1 and C_2 consist of uncrossed edges and so neither is internal to the other. Then by Lemma 2, C_1 and C_2 must live in each other's outer-angle at their common vertex. But this means that the outer-angles are equal, implying that there are three collinear vertices among the vertices of C_1 and C_2 . This contradicts the general position of the vertices of \overline{K}_n . Thus there is no geometric image that shares only a single vertex with C .

Second, suppose C_1 shares an edge of C . Since C is the convex hull of G the third vertex of C_1 is interior to C . By transitivity, this third vertex must also be incident with two geometric images of C and the other geometric image, say C_2 , must share an edge with C_1 . If the third vertex of C_2 is not a vertex of C , then C_2 is a geometric image of C that shares only one vertex with it. We have shown above that this cannot occur. Thus C_2 is a 3-cycle connecting two vertices of C with the third vertex of C_1 . In particular, C and its geometric images create a nonconvex \overline{K}_4 . Since all the edges of this subgraph are geometric images of edges of the convex hull they must be uncrossed. But the minimum number of crossings of a \overline{K}_5 is one, so the addition of another vertex would create a crossing with one of these edges. Thus if there is a geometric copy of C that shares an edge of C , \overline{K}_n is a nonconvex \overline{K}_4 .

We now know that if \overline{K}_n is vertex transitive, nonconvex, and not a nonconvex \overline{K}_4 , then its convex hull is a 3-cycle all of whose geometric copies are disjoint from it (and thus from each other). Suppose that \overline{K}_n contains two such copies C_1 and C_2 . Suppose that C_1 and C_2 are nested with C_2 interior. Then edges from C to C_2 cross the edges of C_1 - a contradiction. Suppose that C_1 and C_2 are not nested. It is straightforward to see that the crossing pattern created by edges between two nested, uncrossed 3-cycles (say C and C_1) is different from the crossing pattern created by edges between two unnested, uncrossed 3-cycles (C_1 and C_2). But these crossing patterns must be geometric images of each other - a contradiction.

In the final case \overline{K}_n has a convex hull C that is a 3-cycle, contains only one geometric image of C , and further, this image is disjoint from C . Thus $n = 6$. Since no edge of C_1 can be crossed, each of the vertices in C must be in the outer-angle of the corresponding vertex of C_1 . Figure 3 depicts this \overline{K}_6 .

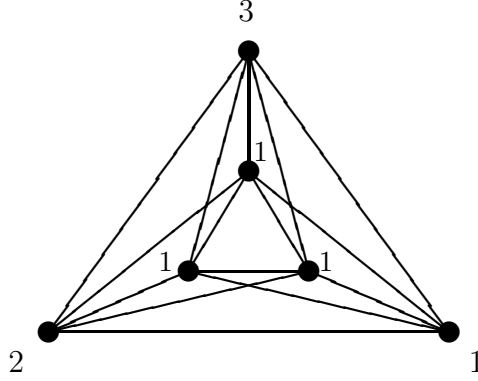


Figure 3: \overline{K}_6 with nested \overline{K}_3 's: $D = 3$

To conclude our proof, we need only show that we can 3-distinguish this particular \overline{K}_6 and any convex \overline{K}_n . The \overline{K}_6 in question can be 3-distinguished as exhibited in Figure 3. When \overline{K}_n is convex, the only uncrossed edges are on the convex hull. Then $H_0 \cong C_n$. By Lemma 1 and the fact that $D(C_n) \leq 3$, we get that $D(\overline{K}_n) \leq D(H_0) = D(C_n) \leq 3$. \square

Lemma 4. If \overline{K}_n contains a vertex orbit that induces a nonconvex \overline{K}_4 , then \overline{K}_n is a nonconvex \overline{K}_4 .

Proof. Suppose that \overline{K}_n is not a nonconvex \overline{K}_4 but that it contains a vertex orbit that induces a nonconvex \overline{K}_4 . Label the vertices in this orbit A, B, C, D , call the induced geometric subgraph \overline{G} , and assume that D is the interior vertex of \overline{G} .

Since \overline{G} is not a spanning subgraph there is some vertex, say Z , in \overline{K}_n that is not in \overline{G} . With respect to \overline{G} there are three different locations for Z . It could be outside \overline{G} but not in an outer-angle of any vertex of $\triangle ABC$; it could be in one of $A^\angle, B^\angle, \text{ or } C^\angle$; or it could be inside the convex hull of \overline{G} . These three possibilities are depicted in Figure 4.

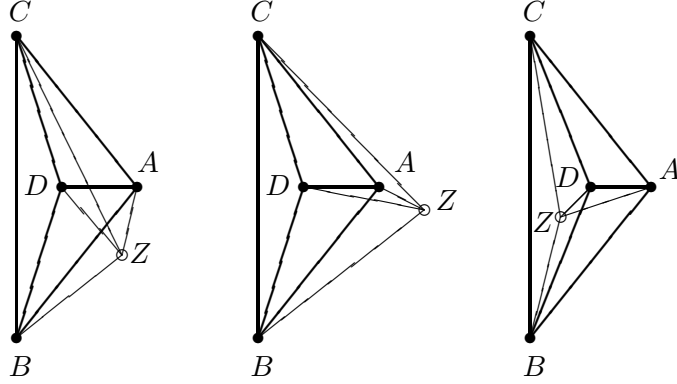


Figure 4: An induced \overline{K}_4 with an additional vertex

We consider the three cases separately.

1. Suppose that Z lives in the exterior of \overline{G} but not in A^\angle , B^\angle , or C^\angle . Notice that the crossing sequence for Z in this case is $0, 0, 1, 2$, while in the other two cases it is $0, 0, 0, 1$. Thus every geometric automorphism of \overline{K}_n must map Z to another vertex that is outside of \overline{G} and not in A^\angle , B^\angle , or C^\angle . Since the vertices of \overline{G} comprise a vertex orbit there is a geometric automorphism, say f , sending D to the convex hull of \overline{G} . Since ZD crosses a single edge of \overline{G} , so must $f(ZD)$. But $f(Z)$ is outside \overline{G} and not in an outer-angle and $f(D)$ is on the convex hull so this is not possible. Therefore no vertex of \overline{K}_n can live outside of \overline{G} without living in an outer-angle of $\triangle ABC$.
2. Suppose that Z lives in an outer-angle of $\triangle ABC$. Without loss of generality suppose $Z \in A^\angle$ and that ZD crosses AB . Since the vertices of \overline{G} comprise a single vertex orbit, there is a geometric automorphism, say f , that takes D to C . Then because ZD crosses AB , $f(ZD)$ must cross $f(AB)$. Since $f(D)$ is on the convex hull of \overline{G} the only ways for $f(ZD)$ to cross an edge would be for $f(Z)$ to be exterior to \overline{G} but not in an outer-angle or for $f(Z)$ to live inside \overline{G} . The former situation was ruled out in case 1. If $f(Z)$ lives inside \overline{G} and $f(ZD)$ crosses $f(AB)$, then $f(AB)$ is an inner edge of \overline{G} . Then either $f(A)$ or $f(B)$ is interior to \overline{G} . That is, either $f(A) = D$ or $f(B) = D$. Then $f(B) = A$ or B and $f(C) = B$ or A . Each of these four choices can be checked separately,

but the results are similar. The following is an examination of one of these choices.

Suppose that $f(A) = D, f(B) = B, f(C) = A$, and $f(D) = C$. Then for $f(ZD)$ to cross $f(AB)$, $f(Z)$ must be inside the 3-cycle formed by $f(A)f(B)f(C)$. Consider the geometric automorphism $g = f^2$: $g(A) = C, g(B) = B, g(C) = D, g(D) = A$. For $g(ZD)$ to cross $g(AB)$ it is necessary that $g(Z)$ live outside \overline{G} on the other side of $g(AB)$ from $g(D)$. But then $g(ZD)$ also crosses an interior edge, which implies that $g(Z)$ lives outside \overline{G} but not in an outer-angle - a contradiction. Thus no vertex of \overline{K}_n can live in an outer-angle of \overline{G} .

3. Suppose that Z lives inside \overline{G} . Without loss of generality assume Z lives in the 3-cycle BCD and that ZA crosses BD . There is a geometric automorphism, say f , so that $f(C) = D$. Note that since $f(C)$ has taken the position interior to the convex hull, both $f(B)$ and $f(D)$ are on the convex hull. In particular, $f(B)f(D)$ is on the convex hull. Since ZA crosses BD , $f(ZA)$ must cross $f(BD)$. But $f(BD)$ is on the convex hull, which implies that $f(Z)$ must live in either D^\angle or B^\angle . But we've proved above that there can be no vertex in an outer-angle of \overline{G} . Thus no vertex of \overline{K}_n can live inside \overline{G} .

Thus if the vertices of \overline{G} are a geometric vertex orbit, \overline{G} is a spanning subgraph and therefore our original graph is a nonconvex \overline{K}_4 . \square

Theorem 1. Every \overline{K}_n that is not a nonconvex \overline{K}_4 can be 3-distinguished.

Proof. Let n be arbitrary. Assume that every \overline{K}_m with $m < n$ that is not a nonconvex \overline{K}_4 can be 3-distinguished.

If \overline{K}_n is vertex transitive it is 3-distinguishable by Lemma 3. If \overline{K}_n is not vertex transitive choose any vertex orbit (which is necessarily a proper subset of the vertices of \overline{K}_n). The graph induced by this orbit is $\overline{G}_1 = \overline{K}_m$ where $m < n$. By Lemma 4, \overline{G}_1 is not a nonconvex \overline{K}_4 and so by the inductive hypothesis it can be 3-distinguished.

Consider the vertices of \overline{K}_n that are not in \overline{G}_1 . They induce $\overline{G}_2 = \overline{K}_{n-m}$. If \overline{G}_2 is not a nonconvex \overline{K}_4 it is 3-distinguishable by the inductive hypothesis. If \overline{G}_2 is a nonconvex \overline{K}_4 it is not a single vertex orbit by Lemma

4. In this case we can 3-distinguish the individual vertex orbits of \overline{G}_2 and thus \overline{G}_2 itself.

Since the vertices in \overline{G}_1 and \overline{G}_2 are distinguished from each other by being in different vertex orbits and the vertices within each of \overline{G}_1 and \overline{G}_2 are 3-distinguished by their induced subgraphs, \overline{K}_n is 3-distinguished. \square

We now know the maximum distinguishing number for each \overline{K}_n . What about the minimum distinguishing number? It is immediate that there is no rigid \overline{K}_n with $n \leq 4$. Since any five points (in general position) in \mathbb{R}^2 contain four in convex position, it must be the case that any \overline{K}_5 contains a convex \overline{K}_4 . It is straightforward to check that no matter where the fifth point is placed, there will still be a nontrivial geometric automorphism.

Theorem 2. For each $n \geq 6$ there exists a rigid \overline{K}_n .

Proof. Figure 5 exhibits a rigid \overline{K}_6 and a rigid \overline{K}_8 . Within each picture the edges of H_0 are shown with solid lines and the edges of H_+ are shown with dashed lines. For \overline{K}_6 it is straightforward to see that H_0 is rigid. To construct a rigid \overline{K}_7 insert a new vertex inside the quadrilateral whose edges are segments of the lines BE, BD, AC , and AF . Within \overline{K}_7 the new H_0 consists of the old H_0 together with an isolated vertex. Therefore the new H_0 is rigid as well.

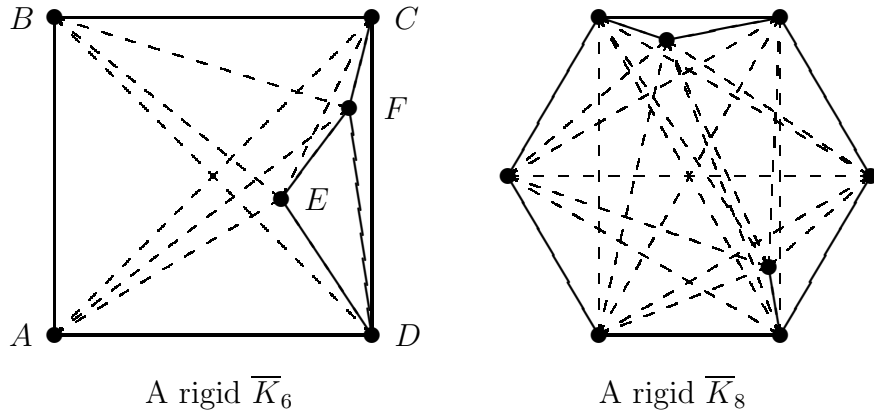


Figure 5: Two rigid \overline{K}_n 's

To construct a rigid \overline{K}_n with $n \geq 8$ we will create an H_0 that consists of a $n - 2$ cycle with degree sequence $3, 3, 2, 3, 2, 2, \dots, 2$. Notice the similarity

to the distinguishing labelling of C_{n-2} with the labels $1, 1, 2, 1, 2, 2, \dots, 2$ that we have previously seen. That is, the degrees of the vertices in H_0 will themselves provide a “labelling” that distinguishes them. Begin with a convex \overline{K}_{n-2} whose vertices are x_1, x_2, \dots, x_{n-2} . Expand this to a \overline{K}_n by adding a vertex u placed in the triangle bordered by x_1x_2, x_1x_3 , and x_2x_{n-2} , and a vertex v placed in the triangle bordered by x_3x_5, x_2x_4 , and ux_4 . In this \overline{K}_n the edges of H_0 are precisely those of the convex hull together with ux_1, ux_2 , and vx_4 . This construction is illustrated in Figure 5 with $n = 8$. Note for ease of drawing the vertices in \overline{K}_8 appear not to be in general position. This is, of course, an illusion. The degrees in H_0 of the vertices on the convex hull are now $3, 3, 2, 3, 2, 2, \dots, 2$ as desired. Thus these vertices can be distinguished from each other. Clearly u and v can be distinguished by their uncrossed adjacencies to the vertices of the convex hull. Thus we can distinguish all vertices of \overline{K}_n without labelling and it is rigid. \square

4 Distinguishing $\overline{K}_{r,n}$

We begin with some elementary observations. Suppose $r = 1$. Since no two edges that share an end point can cross, any permutation of the n -element independent set in $K_{1,n}$ is a geometric automorphism of $\overline{K}_{1,n}$. Consequently if $n \geq 2$, $\text{Aut}(\overline{K}_{1,n}) \cong \text{Aut}(K_{1,n}) \cong S_n$, and $D(\overline{K}_{1,n}) = D(K_{1,n}) = n$.

4.1 Distinguishing $\overline{K}_{2,n}$

As soon as crossings enter the picture the situation gets more complex. This is apparent even with $K_{2,2} \cong C_4$. As we have seen in Example 1, $D(\overline{C}_4)$ will either be 3 or 2 depending on whether or not \overline{C}_4 has a crossing. We next study $\overline{K}_{2,3}$, before giving the general result for $\overline{K}_{2,n}$ where $n \geq 3$.

Theorem 3. The number of uncrossed edges, $|E_0|$, of $\overline{K}_{2,3}$ is 2, 3, 4 or 6. If $|E_0| = 6$, then $D(\overline{K}_{2,3}) = 3$, otherwise $D(\overline{K}_{2,3}) = 2$.

Proof. If the vertices on the convex hull of $\overline{K}_{2,3}$ are in a single color class then there is a unique drawing, and in that drawing $|E_0| = 4$. If the vertices on the convex hull contain vertices from both color classes then the convex hull contains at least two edges and these edges must be in E_0 . Thus in all cases $|E_0| \geq 2$. It cannot be the case that $|E_0| = 5$ since then we would

then conclude the impossible – that $|E_+| = 1$. We will see that each of $|E_0| = 2, 3, 4$ and 6 is realizable.

If $|E_0|=6$, then $\overline{K}_{2,3}$ is a plane graph and $D(\overline{K}_{2,3}) = D(K_{2,3}) = 3$.

If $|E_0| = 4$, then $|E_+| = 2$. H_+ consists of two crossing edges which therefore share no vertex. It is straightforward to see that the complement of H_+ in $\overline{K}_{2,3}$ is $H_0 = P_5$, a path on 5 vertices. If we assign different labels to the end vertices of the P_5 and either of these two labels to the middle vertices we see that $D(H_0) = 2$.

If $|E_0| = 3$, then H_0 cannot be the disjoint union of $K_{1,3}$ and P_1 (denoted by $K_{1,3} \cup P_1$), since if it were, the edges in H_+ would share a common vertex and consequently could not cross. Thus the only possibilities for H_0 are $H_0 = P_3 \cup P_2$ or $H_0 = P_4 \cup P_1$. In each of these cases $D(H_0) = 2$.

If $|E_0| = 2$, then H_+ is either P_5 , the unique tree on five vertices with degrees 3, 2, 1, 1, 1, or C_4 . Since there is no \overline{C}_4 in which every edge is crossed, the last case cannot occur. For the other two cases it is easy to see that $D(H_+) = 2$.

In each of the last three cases we know from Lemma 1 that $D(\overline{K}_{2,3}) \leq 2$. By examining the two different $\overline{K}_{2,2}$'s and all possible ways to complete this to a $\overline{K}_{2,3}$, we can see that there are exactly four different $\overline{K}_{2,3}$'s, and that none of these is rigid. \square

It is immediate that $D(\overline{K}_{2,n}) \leq D(K_{2,n}) = n$ and that the plane drawing of $K_{2,n}$ realizes the distinguishing number n . It may be of some surprise to learn that for each n virtually all possible distinguishing numbers are realizable in some $\overline{K}_{2,n}$.

Theorem 4. For each $n \geq 3$, $\overline{K}_{2,n}$ can realize any distinguishing number between 2 and n inclusive. Further, for each $n \geq 4$, there is a rigid $\overline{K}_{2,n}$.

Proof. For a given j where $2 \leq j \leq n$, we want to construct $\overline{K}_{2,n}$ with $D(\overline{K}_{2,n}) = j$. If $j = n$, the result is just the remark preceding the theorem.

Let $\{u, v\}$ denote the set of red vertices and $\{x_1, x_2, \dots, x_n\}$ denote the set of blue vertices. The following specifies coordinates for the points in our $\overline{K}_{2,n}$. Let $u = (0, 0)$ and $v = (0, 2)$. For $1 \leq i \leq n - j$, let $x_i = (i, 3)$. Finally, for $n - j + 1 \leq i \leq n$, let $x_i = (i, 1)$. Although these points are not in general position, that is only for convenience of presentation. It would be straightforward to modify their positions slightly without affecting the argument that follows.

Note that $E_0 = \{vx_1, ux_{n-j+1}, ux_{n-j+2}, \dots, ux_n\}$. Since $j \geq 2$ and $n \geq 3$, any geometric automorphism must fix both u and v . Since for $1 \leq i \leq n-j$, $cr(ux_i) = n-i$ and $cr(vx_i) = i-1$, any geometric automorphism must fix x_1, x_2, \dots, x_{n-j} . Finally for $n-j+1 \leq i \leq n$, $cr(ux_i) = 0$ and $cr(vx_i) = n-j$. One can easily see that the geometric automorphism group of this realization is the full symmetric group on $\{x_{n-j+1}, x_{n-j+2}, \dots, x_n\}$, and $D(\overline{K}_{2,n}) = j$. Figure 6 illustrates this construction when $n = 10$ and $j = 4$.

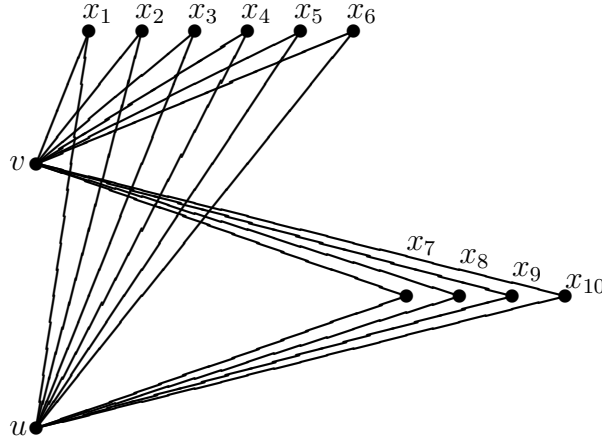


Figure 6: $D(\overline{K}_{2,10}) = 4$

We now construct a rigid $\overline{K}_{2,n}$ when $n \geq 4$. As above, we place u and v on one line and x_1, x_2, \dots, x_{n-1} on another line. For the moment assume that $n = 2t + 1$. Place x_n in the triangular region bounded by the edges vx_1 , ux_t , and ux_{t+1} . We illustrate this in Figure 7 with $\overline{K}_{2,7}$.

The placement of x_n creates different crossing sequences at u and v as well as making all the crossing sequences associated with the x_i 's distinct. Thus there is no nontrivial geometric automorphism. A similar construction works when n is even. \square

4.2 Distinguishing $\overline{K}_{3,3}$

Lemma 5. For any geometric realization of $K_{3,3}$ we have $2 \leq |E_0(\overline{K}_{3,3})| \leq 7$.

Proof. Since $K_{3,3}$ is not planar there must exist at least one pair of crossing edges. Thus $|E_0| \leq 7$. To get the lower bound we look at the convex hull

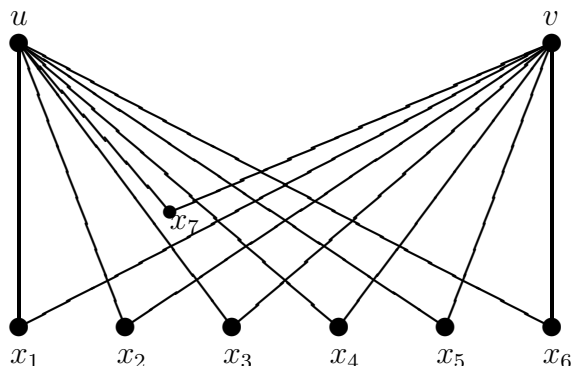


Figure 7: $D(\overline{K}_{2,7}) = 1$

of any realization of $K_{3,3}$. If the convex hull contains both a red and a blue vertex, then it must contain at least two edges, and these edges must be in E_0 . The remaining possibility is that the convex hull consists of three independent (say red) vertices. When we place the first blue vertex in the interior we get three uncrossed edges. When we place the second blue vertex in the interior we have a total of six edges, four uncrossed and one crossed pair. No matter how we place the third blue vertex there will be at least four uncrossed edges. \square

Both bounds in the preceding lemma are sharp.

Theorem 5. $D(\overline{K}_{3,3}) \leq 2$.

One might imagine proving this theorem by listing all geometric realizations of $K_{3,3}$ and finding a 2-distinguishing labelling of each. Writing such a proof is a daunting task. To spare the reader as well as ourselves, we have opted for an argument that, relying on Lemma 1, allows us to examine abstract graphs rather than geometric graphs.

Proof. Note that $\overline{K}_{3,3}$ has nine edges and each edge is in exactly one of E_0 and E_+ . Combining this with Lemma 5 tell us that one of E_0 and E_+ has two, three or four edges. Thus we will enumerate every possible spanning subgraph of $K_{3,3}$ with two, three, or four edges. If H is such a subgraph, then we let H^* denote the spanning subgraph of $K_{3,3}$ such that $E(H^*) = E(K_{3,3}) - E(H)$. That is, H^* is the “complement of H with respect to $K_{3,3}$ ”. For each such pair we will either exhibit a 2-distinguishing labelling of H or a 2-distinguishing

labelling of H^* , or we will show that the particular pair in question cannot be H_0 and H_+ for any $\overline{K}_{3,3}$. Lemma 1 will allow us to immediately conclude the theorem.

Note that although there is a graph automorphism of $K_{3,3}$ that interchanges the two color classes of vertices, there is none that exchanges proper subsets of the color classes. Thus once we distinguish any vertex of $\overline{K}_{3,3}$ from the rest, the color classes can no longer be interchanged, and we only need distinguish vertices within each class.

To make our listing more comprehensible we will specify the two different partite sets using circles for the red vertices and disks for the blue vertices. The cases are organized by $|E(H)|$ and within each case by the sequence of degrees. In any case where we supply a distinguishing labelling we exhibit a figure.

1. $|E(H)| = 2$. The degree sequence (red from left to right, then blue from left to right) is either 2, 0, 0, 1, 1, 0 or 1, 1, 0, 1, 1, 0. See Figure 8.

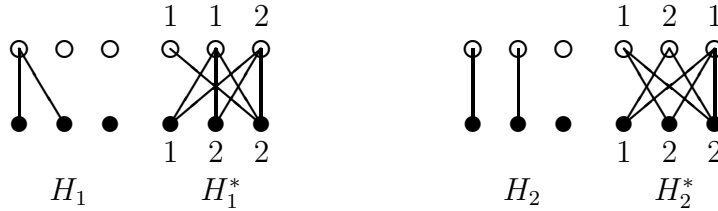


Figure 8:

Our labelling distinguishes H_1^* : The vertex of degree 1 is unique. Then the color classes cannot be interchanged. The two red vertices of degree 3 are labelled 1 and 2 as are the two blue vertices of degree 2. The blue vertex of degree 3 can be given any color.

Our labelling distinguishes H_2^* : The two vertices of degree 3 are given different labels. This then distinguishes the two red vertices of degree 2 from the two blue vertices of degree 2. The vertices in each of these pair are distinguished by different labels.

2. $|E(H)| = 3$ and the degree sequence is 3, 0, 0, 1, 1, 1. In this case we would have $D(H) = D(H^*) = 3$. However, if $H = H_0$, then $H_+ \cong K_{2,3}$.

We have seen that there is no $\overline{K}_{2,3}$ in which every edge is crossed. On the other hand it cannot be the case that $H = H_+$ since edges that are incident with the same vertex cannot cross.

3. $|E(H)| = 3$ and the degree sequence is 1, 1, 1, 1, 1, 1. Here $H^* \cong C_6$ and $D(C_6) = 2$.
4. $|E(H)| = 3$ and the degree sequence is either 2, 1, 0, 2, 1, 0 or 2, 1, 0, 1, 1, 1. See Figure 9.

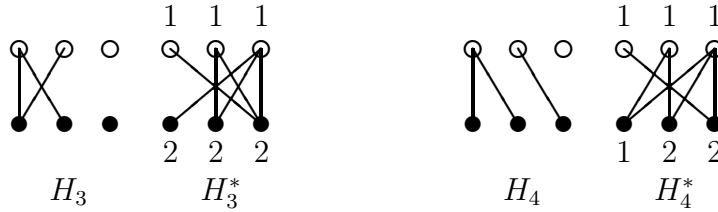


Figure 9:

H_3^* is distinguished since there are exactly two vertices of each possible degree and the labels are different. Our labelling distinguishes H_4^* : The two color classes have different degree sequences and thus cannot be interchanged. The three red vertices all have different degrees and so can be distinguished by any labelling. The first two blue vertices have different labels while the last blue vertex is distinguished by its adjacency to the unique vertex of degree 1.

5. $|E(H)| = 4$ and the degree sequence is either 3, 1, 0, 2, 1, 1 or 2, 2, 0, 2, 2, 0. See Figure 10.

H_5 is distinguished since the two color classes cannot be interchanged, the three red vertices all have different degrees, and the two blue vertices with the same degree have different labels. H_6^* is distinguished since the two vertices of degree 3 have different labels, thereby making the color classes distinguishable, and the two vertices of degree 1 of each color class have different labels.

6. $|E| = 4$ and the degrees are 2, 2, 0, 2, 1, 1. See Figure 11.

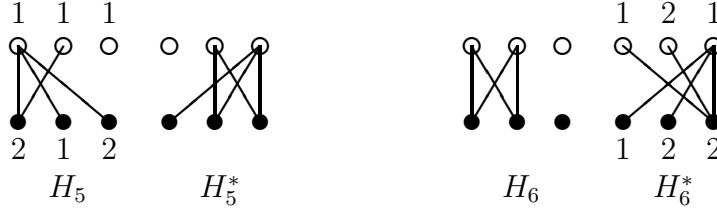


Figure 10:

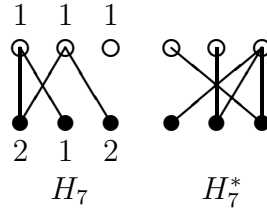


Figure 11:

Our labelling distinguishes H_7 : The degree sequences of the two color classes are different so the color classes cannot be interchanged. The two vertices of degree 1 have different labels, the third blue vertex is distinguished by having degree 2, and the two red vertices of degree 2 are distinguished by the vertex of degree 1 to which they are adjacent.

□

5 Open Questions

We have seen that if \overline{K}_n is convex, then $H_0 = C_n$ and $D(\overline{K}_n) = D(C_n)$. Thus for $n = 3, 4, 5$, convex \overline{K}_n has $D = 3$. However, for $n \geq 6$, convex \overline{K}_n can be 2-distinguished. Figure 12 shows that there is a nonconvex \overline{K}_6 with $D = 3$. Thus when $3 \leq n \leq 6$, $D_g(K_n) = 3$

Question 1. If $n \geq 7$, is $D_g(K_n) = 2$?

Or more generally:

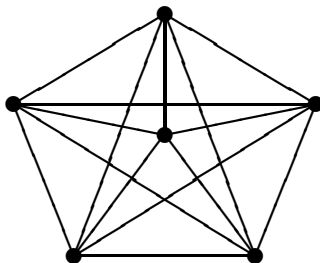


Figure 12: A $\overline{K_6}$ with $D = 3$

Question 2. Does there exist n_0 such that if $n \geq n_0$, then $D_g(K_n) = 2$?

The same question is open for complete bipartite graphs.

Question 3. If $q \geq p \geq 3$, is $D_g(K_{p,q}) = 2$?

We have seen that if $k \leq 2$, then k -connected geometric graphs can have arbitrarily large distinguishing number. There may be a significant contrast for 3-connected graphs.

Question 4. Suppose G is 3-connected. Is there a constant c such that $D_g(G) \leq c$?

In particular the following two questions seem especially tantalizing.

Question 5. Does there exist a 3-connected graph G such that $D_g(G) > 4$?

Question 6. Does there exist an infinite family of 3-connected geometric graphs, say \overline{G}_n , such that $D(\overline{G}_n) > 2$?

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