Isometrically Embedded Graphs

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Abstract

Can an arbitrary graph be embedded in Euclidean space so that the isometry group of its vertex set is precisely its graph automorphism group? This paper gives an affirmative answer, explores the number of dimensions necessary, and classifies the outerplanar graphs that have such an embedding in the plane.

1 Introduction

To prove the existence of a set of points with a specified isometry group, Albertson and Boutin embed a Cayley color graph in Euclidean space so that the vertex isometry group is the graph automorphism group [2]. Such an embedding is called an isometric embedding. In response, Thomassen [11] asked: "Can this be done with an arbitrary graph?" This paper proves an affirmative answer, makes observations on the smallest dimension in which this can be done, and classifies the outerplanar graphs that embed isometrically in the plane.

Embeddings and immersions of graphs are studied for their own sake as well as for use as tools on other problems. Computer scientists look for algorithms that draw graphs "nicely." They want drawings that maximize the number of graph automorphisms that show up as Euclidean isometries. Efficient algorithmic results have been found in the plane for trees [9], outerplanar graphs [10], and planar graphs [7]. Abelson, Hong, and Taylor extended this research to immersions that maximize symmetry in higher dimensions [1]. De Fraysseix created a heuristic to find the automorphism group of a graph by looking at the isometries of a particular embedding [6]. This paper shows that every graph has an embedding in which the vertex isometry group and the graph automorphism group are the same.

The proof of the existence of an isometric embedding is contained in Section 2 along with some basic definitions. It is easy to embed G in $|V|$ dimensions, but this can be trivially lowered to $|V|-1$ dimensions. An obvious question is "What is the smallest dimension in which a given graph can be isometrically embedded?" Section 3 looks at some initial observations. A graph that can be isometrically embedded in the plane is a planar graph. However, the planar graphs K_4 and $K_{2,3}$ each require three dimensions for an isometric embedding. Since subdivisions of these two graphs distinguish planarity from outerplanarity, it is natural to ask "Can all outerplanar graphs be isometrically embedded in the plane?" and "Are outerplanar graphs the only ones that can be isometrically embedded in the plane?" The answers are: "Almost" and "Not quite." In Section 4 we classify the outerplanar graphs that can be isometrically embedded in the plane. Section 5 looks at a few open questions.

2 An Isometric Embedding

A *straight-line drawing* of a graph $G = (V, E)$ is an injective function $F : V \to \mathbb{R}^n$. Represent vertex a by the point $F(a)$ and edge $\{a, b\}$ as the line segment between $F(a)$ and $F(b)$. All graph drawings in this paper shall be straight-line drawings. A graph drawing is called an *embedding* if no two edges intersect at a point unless that point is a vertex to which both edges are incident. We work with both abstract graphs and embedded graphs throughout this paper. When there is the possibility of confusion between the two, we call the vertices and edges of the embedded graph Euclidean vertices and Euclidean edges.

An *isometry* of a set of points S in $\mathbb{R}^n$ is a bijection $\varphi : S \to S$ that preserves distance. This bijection extends naturally to a distance preserving map on the span of S. Thus if S spans an n-dimensional subspace, we may assume that an isometry of S is an isometry of $\mathbb{R}^n$ that restricts to a bijection on S.

Definition 1. A graph $G = (V, E)$ embedded by $F : V \to \mathbb{R}^n$ is said to be *isometrically embedded* if every isometry of $F(V)$ induces an automorphism of G and every automorphism of G induces an isometry of $F(V)$.

As an example, consider a planar embedding of K_4 with maximum symmetry as illustrated in Figure 1. Note that the edges comprising the sides of the equilateral triangle necessarily have length different from the other edges. But because K_4 is edge transitive, all edges in an isometric embedding must have the same length. Thus no planar embedding of K_4 is isometric. However, an isometric embedding of K_4 can be accomplished in $\mathbb{R}^3$ by the vertices and edges of a regular tetrahedron.

The following is the main result.

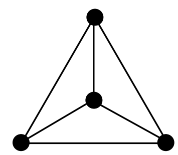

Figure 1: Symmetric, but not isometric, embedding of K_4

Theorem 1. *Every finite graph can be isometrically embedded in finite dimensional Euclidean space.*

The following lemma will make the proof of the theorem easy.

Lemma 1. *Let P be the set of all unordered pairs of integers from the set $\{1, \cdots, n\}$. Given a partition of P into blocks, there exist points $X_1, \cdots, X_n$ in $\mathbb{R}^n$ so that*

a) each point is on the unit sphere;

b) the vectors $\vec{X}_1, \cdots, \vec{X}_n$ are linearly independent;

c) the j^{th} coordinate of X_i is zero if $i < j$;

d) $d(X_i, X_j) = d(X_k, X_\ell)$ if and only if $\{i, j\}$ and $\{k, \ell\}$ are contained in the same block of the partition.

Both the statement and the proof of this lemma are minor adaptations of Lemma 1 from [2]. Notice that the endpoints of the standard basis vectors in $\mathbb{R}^n$ fulfill conditions a), b) and c). Albertson and Boutin use the Implicit Function Theorem to show that, given a partition of 2-subsets of an n-set, there is a perturbation of the endpoints of the standard basis vectors in which pairwise distance is determined by partition block, and conditions a) through c) are maintained. See [2] for details. Lovász [8] remarks that work of Deza and Laurent [5] can be used to obtain a similar result.

Proof of Theorem: An edge of a graph is an unordered pairs of distinct vertices. It is useful here to call an unordered pair of vertices that is not an edge a *non-edge*. Since graph automorphisms preserve edges and non-edges, the term *non-edge orbit* is both well-defined and useful in our circumstances.

To begin the proof, let G be a graph with n vertices. Partition the set of unordered pairs of distinct vertices into $\mathrm{Aut}(G)$-orbits. Each orbit is either an edge orbit or a non-edge orbit. Using Lemma 1, there exist n points in $\mathbb{R}^n$, which we label by vertex labels of G, whose pairwise distances are distinguished by Aut(G)-orbit. That is, $d(x,y) = d(u,v)$ if and only if there exists $\theta \in$ Aut(G) so that $\{\theta(x), \theta(y)\} = \{u, v\}$. Consider these points Euclidean vertices and add Euclidean edges as appropriate to obtain G. This is an embedding since the n vectors determined by the Euclidean vertices are linearly independent.

Aut(G) acts on the vertices of this embedding by acting on the associated vertex labels. This action preserves pairwise distance since it preserves Aut(G)-orbits. Thus each graph automorphism induces an isometry on the vertices.

Suppose σ is an isometry of the vertices of the embedded graph. Then for any two Euclidean vertices x and y, $d(x,y) = d(\sigma(x), \sigma(y))$. Since pairwise distance is distinguished by Aut(G)-orbit, $\{\sigma(x), \sigma(y)\}$ must be in the same orbit as $\{x, y\}$. Thus $\{x, y\}$ is an edge if and only if $\{\sigma(x), \sigma(y)\}$ is an edge. Then σ induces a graph automorphism of G.

Thus we have an isometric embedding of G in $\mathbb{R}^n$. $\square$

3 Isometric Embedding Dimension

It is natural to ask "What is the smallest dimension in which a given graph G can be isometrically embedded?" Call this the *isometric embedding dimension* of the graph and denote it $\bar{\delta}(G)$. The proof of Theorem 1 shows that $\bar{\delta}(G) \leq |V|$ but this can be lowered to dimension $|V| - 1$ by considering the span of the Euclidean vertices.

Proposition 1. *If* Aut(G) *produces every possible permutation on a set of n vertices of G then $\bar{\delta}(G) \geq n - 1$.*

This proof is similar to the one below and is left to the reader.

Proposition 1 verifies that K_4 has isometric embedding dimension at least 3. Since an isometric embedding of K_4 is provided by the vertices and edges of a regular tetrahedron, $\bar{\delta}(K_4) = 3$. This proposition also tells us that since all permutations of the leaves of $K_{1,n}$ occur as graph automorphisms, its isometric embedding dimension is at least $n - 1$. Thus isometry dimension is unbounded, even when the graphs under consideration are trees.

Proposition 2. *If W is a set of points spanning r dimensions and V is a set of points spanning s dimensions, and if for each $v \in V$ there is a fixed distance d so that each point of W is distance d from v, then $W \cup V$ spans at least $r + s$ dimensions.*

Proof. Notice that to span s dimensions V must contain at least $s+1$ points. Let $\{v_0, \cdots, v_s\}$ be a set of $s+1$ points of V so that $\{\vec{v}_0 - \vec{v}_i\}_{i=1}^{s}$ is a linearly independent set. Since each point of W is at distance d_0

from v_0, v_0 is the center of a sphere on which all points of W reside. Then if v_0 is inside the span of W, it is the center of an $(r-1)$-sphere which we denote S_0. If v_1 is also within the span of W then it too is the center of an $(r-1)$-sphere which we denote S_1. If two $(r-1)$-spheres are distinct then their intersection is either empty, a single point, or an $(r-2)$-sphere [3]. But $S_0 \cap S_1$ contains W and therefore spans r dimensions, which is a contradiction. Then at least one of v_0 and v_1 lies outside the span of W; without loss of generality we may assume that v_1 does. In particular this shows that $\dim(\text{span}(W \cup \{v_0, v_1\})) \geq r+1$. Thus we can show that if v_0 is inside the span of W none of $v_1, \cdots, v_s$ is, and therefore $\dim(\text{span}(W \cup V)) \geq r+s$. □

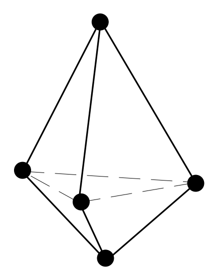

Figure 2: An isometric embedding of $K_{2,3}$

Proposition 2 verifies that $K_{2,3}$ requires at least 3 dimensions for an isometric embedding. Such an embedding is realized by the vertices and a subset of the edges of a double triangular pyramid, as illustrated in Figure 2. Thus $\bar{\delta}(K_{2,3}) = 3$.

A related parameter, the isometry dimension of a group Γ, denoted $\delta(\Gamma)$, is defined to be the smallest integer n for which there exists a finite set of points in $\mathbb{R}^n$ whose isometry group is Γ [2]. Clearly for any graph G, $\delta(\text{Aut}(G)) \leq \bar{\delta}(G)$. However, consider the graph consisting of C_4 with opposite pairs of vertices connected by paths of length three, as illustrated in Figure 3. This is a planar graph with automorphism group D_4. The isometry dimension of D_4 is 2 but the isometric embedding dimension of the graph is 3. Thus $\delta(\text{Aut}(G))$ is in general not equal to $\bar{\delta}(G)$.

4 Planar Isometric Embedding

Which graphs can be isometrically embedded in the Euclidean plane? If a graph is embedded in the plane it must be, by definition, a planar

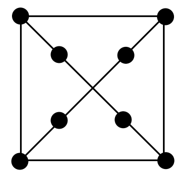

Figure 3: Planar, but not isometrically embeddable in the plane

graph. If it is isometrically embedded, its automorphism group acts by isometries on the Euclidean plane. That means, if non-trivial, the automorphism group must "act like" a finite cyclic or dihedral group.

Let's look at an example of what we don't want. The automorphism group of $K_{2,3}$ is $\mathbb{Z}_2 \times S_3 \cong D_6$. Under the isometries of D_6 a point of the Euclidean plane has a stabilizer equal to one of: the trivial group, a subgroup generated by a reflection, or D_6 itself. However, under graph automorphisms the vertices in the smaller vertex color class of $K_{2,3}$ have S_3 stabilizers. Thus though $\mathrm{Aut}(G) \cong D_6$, it cannot simultaneously act as isometries of the Euclidean plane and as automorphisms of $K_{2,3}$.

The following definition captures what it means for an automorphism group to "act like" $\mathbb{Z}_n$ or D_n and rules out the situation described above. Note that $\mathbb{Z}_2 \cong D_1$ (though their actions on the plane are different). Thus we assume $n \geq 1$ for D_n and $n \geq 3$ for $\mathbb{Z}_n$.

Definition 2. Let G be a graph. For $n \geq 3$, we say G has *$\mathbb{Z}_n$-symmetry* if $\mathrm{Aut}(G) \cong \mathbb{Z}_n$, at most one vertex has stabilizer $\mathrm{Aut}(G)$, and every other vertex has trivial stabilizer. For $n \geq 2$, we say G has *D_n-symmetry* if $\mathrm{Aut}(G) \cong D_n$, at most one vertex has stabilizer $\mathrm{Aut}(G)$, and every other vertex either has trivial stabilizer or stabilizer generated by a reflection. We say G has *D_1-symmetry* if $\mathrm{Aut}(G) \cong D_1$ and the set of fixed vertices induces a simple path.

Though planarity and trivial, $\mathbb{Z}_n$, or D_n-symmetry are necessary for a graph to be isometrically embedded in the plane, they are not sufficient. Our previous example shown in Figure 3 is a planar graph with D_4-symmetry that cannot be isometrically embedded in $\mathbb{R}^2$.

Since neither K_4 nor $K_{2,3}$ can be isometrically embedded in the plane and subdivisions of these two graphs distinguish planarity from outerplanarity, it is natural to ask "Are outerplanar graphs the only ones that can be isometrically embedded in the plane?" and "Can all outerplanar graphs with trivial, $\mathbb{Z}_n$ or D_n-symmetry be isometrically embedded in the plane?"

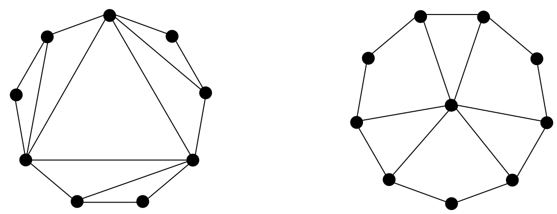

Figure 4: $\mathbb{Z}_3$-symmetry and D_3-symmetry.

The answer to the first question is negative since the n-wheel is not outerplanar but it embeds isometrically in the plane as the star of a regular n-gon. The second question also has a negative answer as demonstrated by the following example. Let G be a 3-cycle with three simple paths attached to each vertex: one of length one, one of length two, and one of length three. Then G is outerplanar and has D_3-symmetry. However, an isometric embedding of G in $\mathbb{R}^2$ would necessarily embed the 3-cycle of G as an equilateral triangle and each simple path would lie on the perpendicular bisector of the opposite side of the triangle. It is not possible to embed all three paths on the same perpendicular bisector. So G is cannot be isometrically embedded in the plane.

Though outerplanarity is not sufficient for an isometric embedding, biconnected outerplanarity is.

Theorem 2. *Every biconnected outerplanar graph can be isometrically embedded in* $\mathbb{R}^2$.

Proof. We may assume that G has three or more vertices.

By [4] every biconnected outerplanar graph on three or more vertices contains a unique Hamiltonian cycle drawn on its exterior face. Since the Hamiltonian cycle is unique, every graph automorphism preserves it. Thus $\mathrm{Aut}(G)$ is a subgroup of the automorphisms of $C_{|V|}$.

Suppose that G has non-trivial symmetry. Since every automorphism of a cycle is either a rotation or a reflection with appropriate vertex stabilizers G has $\mathbb{Z}_n$ or D_n-symmetry. The vertex orbits in a cycle with $\mathbb{Z}_n$-symmetry have cardinality n; the vertex orbits in a cycle with D_n-symmetry have cardinality n or $2n$. Thus G contains kn vertices for some positive integer k. Choose a direction around the outside face and label the vertices $v_1, \cdots, v_{kn}$ in the order encountered.

Place kn Euclidean vertices, with vertex labels from G, in order, symmetrically around the unit circle. This point set has isometry group D_{kn}. Assume that v_1 is in a vertex orbit of minimal cardinality in G

and, for some reasonably small ϵ, dilate the Euclidean vertices associated to the orbit of v_1 to a distance of $1+\epsilon$ from the origin. See the left-hand graph in Figure 5 for an example of the result. Notice that the only remaining isometries are the isometries of this orbit. Thus the point set now has isometry group D_n unless both the minimal orbit and the graph contain $2n$ vertices. In this case the point set has isometry group D_{2n}. Thus we have attained our desired isometry group unless either 1) G has $\mathbb{Z}_n$-symmetry or 2) G has D_n-symmetry and a single vertex orbit of size $2n$.

Figure 5: Euclidean vertices with D_5, $\mathbb{Z}_5$ isometry group respectively

1) Suppose that G has $\mathbb{Z}_n$-symmetry. Note that there is no graph on n vertices with $\mathbb{Z}_n$-symmetry. Then G has kn vertices where $k \geq 2$. Rotate the Euclidean vertex v_2, and its graph automorphic images, by a small angle α. Now there can be no reflection over v_1 because the angle between $\vec{v}_1$ and $\vec{v}_2$ is different from the angle between $\vec{v}_1$ and $\vec{v}_n$. Also there can be no reflection between v_1 and v_{k+1} (the image of v_1 under a rotation through an angle of $\frac{2\pi}{n}$) because the angle between $\vec{v}_1$ and $\vec{v}_2$ is different from the angle between $\vec{v}_k$ and $\vec{v}_{k+1}$. Thus we have broken the remaining reflectional symmetry and the modified Euclidean vertex set has isometry group $\mathbb{Z}_n$ as desired. See the right-hand graph in Figure 5 for an example of the result.

2) Suppose G has D_n-symmetry with a single vertex orbit of size $2n$. Choose two distinct angles α_1 and α_2 so that $\alpha_1+\alpha_2 = \frac{2\pi}{n}$. Rotate the Euclidean vertices so that for all $i = 1, \cdots, n$ the angle between $\vec{v}_{2i-1}$ and $\vec{v}_{2i}$ is α_1 and the angle between $\vec{v}_{2i}$ and $\vec{v}_{2i+1 \,(\mathrm{mod}\ 2n)}$ is α_2. This set of Euclidean vertices now has isometry group D_n as required.

Suppose that G has trivial symmetry. As before, start by placing Euclidean vertices symmetrically around the unit circle. For suitably small distinct ϵ_1, ϵ_2, pull one vertex out to a distance of $1+\epsilon_1$ from the origin and another vertex out to distance $1+\epsilon_2$. With $\epsilon_1 \neq \epsilon_2$ we have broken both the rotational and the reflectional symmetries, obtaining a point set with trivial isometry group.

Add edges as appropriate to create an outerplanar drawing of G. $\square$

Consider the more general case of a connected outerplanar graph. We know it can be embedded isometrically in $\mathbb{R}^n$ only if it has trivial, $\mathbb{Z}_n$ or D_n-symmetry, but what other restrictions apply? We look at two propositions and then state a theorem.

Proposition 3. *If G is a connected outerplanar graph with trivial automorphism group then it can be isometrically embedded in $\mathbb{R}^2$.*

The proof is the same as that for the biconnected case.

Proposition 4. *If G is a connected outerplanar graph with $\mathbb{Z}_n$-symmetry ($n \geq 3$) then it can be isometrically embedded in $\mathbb{R}^2$.*

Proof. The strategy for the proof that follows is to show that having $\mathbb{Z}_n$-symmetry is a strong condition that rules out many of the cases we would otherwise need to study. First we see that the block-cut-vertex tree for the graph cannot have a central cut-vertex. Then we see that the central biconnected block must contain kn vertices for some positive integer k, and that $k \neq 1$. This leaves us with the case that the central biconnected block contains kn vertices where $k \geq 2$; this case is easy to construct using methods from the proof of Theorem 2.

We have already seen this proposition is true when G is biconnected. If G is connected but not biconnected then it has a non-trivial block-cut-vertex tree. A simple argument shows that the center of a block-cut-vertex tree is a single vertex - either a B-vertex representing a central block in the graph or a C-vertex representing a central cut-vertex in the graph. The center of a graph is invariant under every graph automorphism. If the block-cut-vertex tree has a C-vertex at its center then the central cut-vertex of G is fixed by every automorphism of G. If the block-cut-vertex tree has a B-vertex as its center then the central block is invariant under every automorphism of G. That is, each automorphism of G induces an automorphism of the central block. The connected components attached to the central cut-vertex or attached to the center block at its cut-vertices will be called *branches*. The union of all branches at a cut-vertex is called the *branching structure* at that vertex. Since G has $\mathbb{Z}_n$-symmetry, each branch has an orbit of size n. Suppose G has a central cut-vertex. Since a cut-vertex does nothing to reduce symmetry, every permutation of the n branches in a branch orbit occurs as a graph automorphism. Thus $S_n \leq \mathrm{Aut}(G)$. However, $S_n \leq \mathbb{Z}_n$ only for $n = 1, 2$, so this case does not occur.

Thus G has a central block B. A careful examination shows that the $\mathbb{Z}_n$-symmetry requires that B has kn vertices for some positive integer

k and that no branching structure has non-trivial symmetry that fixes the cut-vertex it shares with B.

Suppose that B has n vertices. Then B is vertex transitive and the branching structures are all isomorphic. We can then define a graph automorphism of G that is a reflection on the n-cycle of B and extends to the branching structure in the obvious way. Since $n \geq 3$ this reflection is distinct from any of the rotations in $\mathrm{Aut}(G)$. Thus $\mathbb{Z}_n$ is a proper subgroup of $\mathrm{Aut}(G)$ - a contradiction. Thus if $\mathrm{Aut}(G)$ has $\mathbb{Z}_n$-symmetry and G has a central biconnected block this block has more than n vertices.

Then $|V(B)| = kn$ where $k \geq 2$. As shown in the proof of Theorem 2, we can place kn vertices in $\mathbb{R}^2$ with isometry group $\mathbb{Z}_n$ and add Euclidean edges as appropriate to obtain an embedding of B. For each vertex orbit of B attach a planar drawing of the appropriate branching structure to an associated Euclidean vertex. Add the orbits of these branching structures under the isometries of the Euclidean vertices for B. (Clearly we may scale the branches so that they do not intersect each other.) Since the isometry group of the point set for B is $\mathrm{Aut}(G)$ and the new Euclidean vertices are added in a way that do not disturb this symmetry, the isometry group of the completed set of Euclidean vertices is $\mathrm{Aut}(G)$. Thus we have our isometric embedding. □

We can perform a careful case-by-case analysis using methods similar to those above to determine precisely which connected outerplanar graphs with D_n-symmetry can be isometrically embedded in the plane. Putting these results together with the propositions above we get the following theorem.

Theorem 3. *A connected outerplanar graph G can be isometrically embedded in $\mathbb{R}^2$ if and only if*

1. *G has trivial automorphism group, or*
2. *G has $\mathbb{Z}_n$-symmetry, or*
3. *G has D_n-symmetry and*
   (a) *G is biconnected, or*
   (b) *G has a central biconnected block B and at any cut-vertex of B that is fixed by a reflection there is no branch that is not a simple path and at most two branches that are simple paths, or*
   (c) *G has a central cut-edge E, $n = 1$, and if the reflection fixes E then there is at most one branch at each vertex of E*

that has an edge that is both incident to E and fixed by the reflection, or

(d) *G has a central cut-vertex, $n = 3$, and the branches consist of precisely three simple paths of the same length, or*

(e) *G has a central cut-vertex, $n = 1$, and the branches consist of precisely two simple paths of the same length ℓ_1 and at most two other simple paths of lengths ℓ_2 and ℓ_3, where ℓ_1, ℓ_2, ℓ_3 are distinct.*

5 Open Questions

Which planar graphs with trivial, $\mathbb{Z}_n$ or D_n-symmetry can be isometrically embedded in $\mathbb{R}^2$? Any outerplanar graph G that can be embedded isometrically in $\mathbb{R}^2$ can be extended to a planar graph that also has such an embedding in the following way. Let F be any face of G. Add a central vertex to F with edges to a subset of the vertices of F that is invariant under any automorphism of G under which F itself is invariant. Add the orbits of this "star" under the action of $\mathrm{Aut}(G)$. The result is clearly a planar (not outerplanar) graph that embeds in $\mathbb{R}^2$ isometrically. Further for some outerplanar graphs that have too much symmetry to have trivial, or $\mathbb{Z}_n$, or D_n-symmetry, we can use the selective addition of stars to break symmetry and obtain a planar graph with an isometric embedding. This leads to the following questions:

• Is every planar graph that embeds isometrically in $\mathbb{R}^2$ a "star extension" of an outerplanar graph? No. Which ones are?

• Can every graph that has trivial, $\mathbb{Z}_n$ or D_n-symmetry and is star extension of an outerplanar graph be isometrically embedded in $\mathbb{R}^2$? No. Which ones can?

Are there bounds on isometric embedding dimension? While there is no bound for general planar or outerplanar graphs, we've proved that biconnected outerplanar graphs have isometric embedding dimension at most 2. This leads to the following question(s):

• Is there a bound on the isometric embedding dimension of biconnected planar graphs? Triconnected planar graphs?

Acknowledgement

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