Realizing Finite Groups in Euclidean Space

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Michael O. Albertson [1]
Department of Mathematics, Smith College, Northampton MA 01063
albertson@math.smith.edu

Debra L. Boutin
Department of Mathematics, Hamilton College, Clinton NY 13323
dboutin@hamilton.edu

Abstract

A set of points W in Euclidean space is said to *realize* the finite group G if the isometry group of W is isomorphic to G. We show that every finite group G can be realized by a finite subset of some $\mathbb{R}^n$, with $n < |G|$. The minimum dimension of a Euclidean space in which G can be realized is called its *isometry dimension*. We discuss the isometry dimension of small groups and offer a number of open questions.

1 Introduction

An object X is said to *realize* the group G if $\mathrm{Aut}(X) \cong G$. Here $\mathrm{Aut}(X)$ is the set of bijections from X to itself that preserve whatever is essential about X. A class of objects $\mathcal{C}$ is said to be *(finitely) universal* if given a finite group G, there exists an $X \in \mathcal{C}$ that realizes G. The existence of a universal class is not immediate. Cayley's Theorem, that every group acts as a permutation group on itself, is a precursor of such a universality result. It is a precursor because typically G is isomorphic to a proper subgroup of $\mathrm{Aut}(G)$.

Cayley graphs were the first class identified as universal [2, 7]. The *full Cayley graph* of G is a directed graph whose vertices correspond to the elements of G. For every group element h the vertex g_1 is joined to the vertex g_2 by a directed edge labeled h precisely when $g_1 h = g_2$. Combinatorists call

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this the *full Cayley color graph* thinking of the edge labels as colors. It is common to construct a proper subgraph by inserting only the directed edges that correspond to a (minimal) set of group generators. Cayley constructed these graphs to aid in group visualization and computation, but did not address the issue of their automorphism groups. That result had to wait over sixty years.

In the first graph theory text König asked the (in retrospect) natural question "Which groups are realized by graphs?" [6]. Frucht showed that graphs are universal by first showing that the class of full Cayley color graphs are universal and then by showing how to represent the colored directed edges by graph gadgets [5, 7]. Since Frucht's result there has been a steady trickle of universality results. Here is a sample of what is known: topological spaces, algebraic number fields, and 3-colorable graphs are universal [3, 4, 9]; while trees, planar graphs, and groups are not.

Our goal in this paper is to show that subsets of Euclidean space are universal. That is, given a finite group G, there exists $W \subset \mathbb{R}^n$ for some n, such that G is isomorphic to the isometry group of W. An *isometry* of W is a bijection that preserves Euclidean distance. We do not assume that an isometry of W extends to an isometry of the entire space, though all of those that we construct do extend. Since any group of order n is isomorphic to a subgroup of S_n and S_n can act by permutations on the standard basis vectors in $\mathbb{R}^n$, it is straightforward to see that a group of order n can act by isometry on $\mathbb{R}^n$. The content here is to show that there are subsets whose full isometry group is isomorphic to G.

We prove our result in Section 2 by first showing that for a group G of order n, there exists a set $W \subset \mathbb{R}^n$, consisting of $n + \binom{n}{2}$ points, whose isometry group is isomorphic to G; a corollary shows that we could adjust W slightly to find $W' \subset \mathbb{R}^{n-1}$ that realizes G. The proof has several good features. The set W is finite; the points in W are nicely placed; and there is a strong analogy with the full Cayley graph. Furthermore, each isometry of W extends to an isometry of $\mathbb{R}^n$, and the proof contains an elegant use of the Implict Function Theorem. However, some aspects of the proof are less than ideal. We show that W exists but do not construct it; the W we "find" often has more points than is necessary; and $\mathbb{R}^{n-1}$ often has larger dimension than is necessary. For example the set of k vertices of a regular k-gon in $\mathbb{R}^2$ realizes the dihedral group D_k. Our proof gives $2k^2 + k$ points in $\mathbb{R}^{2k-1}$.

In Section 3 we examine the natural question of the *isometry dimension* of a group, $\{$*viz.* the minimum d such that G is realized as the isometry group of a point set in $\mathbb{R}^d\}$. With the exception of the quaternions we determine the isometry dimension of all groups of order eight or less. All we know about the isometry dimension of the quaternions is that it is at least 4. In Section 4 we list some open questions.

2 The Proof

Theorem 1. Let G be a group of order n. There exists $W \subset \mathbb{R}^n$ such that $|W| = n + \binom{n}{2}$, the isometry group of W is isomorphic to G, and the elements of G naturally act as isometries on $\mathbb{R}^n$.

Proof. We imagine labeling n equidistant points on the unit sphere in $\mathbb{R}^n$ by elements of G. We allow G to act on these points by right action on the labels and then we sort the $\binom{n}{2}$ edges (pairs of vertices) into G-edge orbits. We will change the lengths of these edges so that two edges have the same length if and only if they are in the same G-edge orbit. We invoke a lemma that guarantees the existence of points with appropriate pairwise distances. We then insert orientation points and midpoints to eliminate or allow certain edge inversions. This construction is analogous to that of the full Cayley graph. Here the color or label on each edge is replaced by a distinct edge length, and the direction of the edge is replaced by an orientation point or a midpoint.

Suppose the elements of G are $g_1, g_2, \ldots, g_n$ where g_1 is the identity which will usually be denoted by 1. For $k = 2, \ldots, n$ we wish to assign a distance $|g_k|$ so that $|g_i| = |g_j|$ if and only if either $g_j = g_i$ or $g_j = g_i^{-1}$.

Lemma 1. There exist distances $|g_2|, \ldots, |g_n|$ so that $|g_i| = |g_j|$ if and only if either $g_j = g_i$ or $g_j = g_i^{-1}$ and points $X_1, \ldots, X_n$ in $\mathbb{R}^n$ such that $X_1, \ldots, X_n$ form a basis for $\mathbb{R}^n$, and

1. the j^{th} coordinate of X_i is zero if $i < j$;
2. each point X_i is on the unit sphere; and
3. the distance between X_i and X_j is $|g_i g_j^{-1}|$.

We defer the proof of the lemma.

We call $X_1, \ldots, X_n$, the points whose existence is guaranteed by the lemma, *group points*. We label X_i by g_i and commonly identify these points by their labels. Define a right action of G on the group points by the right permutation action of G on the labels. Specifically $X_i g = X_j$ precisely if $g_i g = g_j$. Notice that if we apply g to the edge $[g_i, g_j]$ we get $[g_i g, g_j g]$, an edge of length $|(g_i g)(g_j g)^{-1}| = |g_i g_j^{-1}|$. Thus G's action on the group points is an isometry. Similarly, by construction of the X_i, if two edges have the same length they are in the same G-edge orbit. Since $\{X_1, \ldots, X_n\}$ forms a basis for $\mathbb{R}^n$, if we use $\vec{g}_i$ to denote the vector from the origin to X_i every point y of $\mathbb{R}^n$ can be written uniquely as $y = a_1\vec{g}_1 + \cdots + a_n\vec{g}_n$. The action of G extends linearly to all of $\mathbb{R}^n$ by $yg = a_1\vec{g_1 g} + \cdots + a_n\vec{g_n g}$.

We next show that this extension of the G-action to $\mathbb{R}^n$ is an isometry.

$$dist^2(g_i, g_j) = \|g_i - g_j\|^2 = \vec{g}_i \cdot \vec{g}_i - 2\vec{g}_i \cdot \vec{g}_j + \vec{g}_j \cdot \vec{g}_j.$$

Similarly,

$$dist^2(g_i g, g_j g) = \|g_i g - g_j g\|^2 = \vec{g_i g} \cdot \vec{g_i g} - 2\vec{g_i g} \cdot \vec{g_j g} + \vec{g_j g} \cdot \vec{g_j g}.$$

Since g acts by isometries on the group points,

$$\vec{g}_i \cdot \vec{g}_i - 2\vec{g}_i \cdot \vec{g}_j + \vec{g}_j \cdot \vec{g}_j = \vec{g_i g} \cdot \vec{g_i g} - 2\vec{g_i g} \cdot \vec{g_j g} + \vec{g_j g} \cdot \vec{g_j g}.$$

Every group point is on the unit sphere so $\vec{g_i g} \cdot \vec{g_i g} = \vec{g}_i \cdot \vec{g}_i = 1$ and we may conclude that $\vec{g}_i \cdot \vec{g}_j = \vec{g_i g} \cdot \vec{g_j g}$. Writing the points of $\mathbb{R}^n$ as linear combinations of the basis $\{\vec{g}_1, \ldots \vec{g}_n\}$ and using the identities above, it is straightforward to verify that the action of G on $\mathbb{R}^n$ is an isometry.

Now that we have edge lengths to serve as our edge labels, we almost have a geometric analogue of the full Cayley graph of G. Only the orientation on the edges is missing. To get orientation add $\binom{n}{2}$ points on the interior of the unit sphere in the following way: Choose one edge $[g_i, g_j]$ in each G-edge orbit. If no element of G inverts the edge add the *orientation point* $g_i + \frac{1}{\sqrt{2}}(g_j - g_i)$ and all its images under G (i.e., $g_i g + \frac{1}{\sqrt{2}}(g_j g - g_i g)$ for each $g \in G$). Each of these points is $\frac{1}{\sqrt{2}}^{th}$ along the line segment from $g_i g$ to $g_j g$. If there is an element of G that inverts the edge add the *midpoint* $g_i + \frac{1}{2}(g_j - g_i)$ and all its images under G.

We now have a set of $n + \binom{n}{2}$ group points, orientation points and midpoints. Call this the set of Cayley points. Since G acts by isometries on all of $\mathbb{R}^n$ it acts by isometries on the Cayley points.

We wish to show that any isometry of these Cayley points is an isometry already given by the action of G. The main facts we need are that edges between group points have equal length if and only if they are in the same G-edge orbit and that the G-stabilizer of every group point is trivial.

Let σ be an isometry of the Cayley points. The group point associated with the identity element 1 is mapped by σ to another Cayley point, say g. We would like to show that for every point g_i, $g_i\sigma = g_i g$. Since σ is an isometry, $[1, g_i]$ and $[1\sigma, g_i\sigma] = [g, g_i\sigma]$ have the same length. Thus they are images of each other under some element $h \in G$. Then either $[g, g_i\sigma] = [1, g_i]h$ or $[g, g_i\sigma] = [g_i, 1]h$. If $[g, g_i\sigma] = [1, g_i]h$, we can conclude that $g = 1h$ and thus that $g_i\sigma = g_i h = g_i g$. If the edge $[1, g_i]$ has an orientation point, the orientation must be preserved under the isometry σ and so $[g, g_i\sigma] = [1, g_i]h$. If the edge $[1, g_i]$ has a midpoint, then there is some $k \in G$ for which $[1, g_i]k = [g_i, 1]$ which tells us that $k = g_i$ and $g_i^2 = 1$. Then if $[g, g_i\sigma] = [g_i, 1]h$ we have $g = g_i h$ and still $g_i\sigma = h = g_i g$.

Thus as isometries $\sigma = g$ and the group of isometries of the Cayley points is precisely G.

$\square$

Now we will restate and prove Lemma 1.

Lemma 1. There exist distances $|g_2|, \ldots, |g_n|$ so that $|g_i| = |g_j|$ if and only if either $g_j = g_i$ or $g_j = g_i^{-1}$ and points $X_1, \ldots, X_n$ in $\mathbb{R}^n$ such that $X_1, \ldots, X_n$ form a basis for $\mathbb{R}^n$, and

1. the j^{th} coordinate of X_i is zero if $i < j$;
2. each point X_i is on the unit sphere; and
3. the distance between X_i and X_j is $|g_i g_j^{-1}|$.

Proof. For each t with $1 \leq t \leq n^2$ there exist unique integers s and r where $1 \leq s, r \leq n$ so that $t = (s-1)n + r$. Let $\bar{X} = (X_1, \ldots, X_n) = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{nn})$ and $\bar{\delta} = (\delta_{12}, \ldots, \delta_{(n-1)n})$.

Consider the following functions whose domains are $\mathbb{R}^{n^2+\binom{n}{2}}$. The domain variables are x_{ij} for $1 \leq i, j \leq n$ and δ_{ij} for $1 \leq i < j \leq n$.

- $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = x_{sr}$ if $s < r$,
- $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = x_{s1}^2 + \cdots + x_{sn}^2 - 1$ if $s = r$, and
- $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = (x_{s1} - x_{r1})^2 + \cdots + (x_{sn} - x_{rn})^2 - \delta_{rs}^2$ if $s > r$.

For $s < r$, $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = 0$ precisely when Condition 1 of the lemma holds. For $s = r$, $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = 0$ precisely when Condition 2 of the lemma holds. For $s > r$, $f_{(s-1)n+r}(\bar{X}, \bar{\delta}) = 0$ precisely when the distance from X_i to X_j is δ_{ij}. Thus, if we assign $\delta_{ij} = |g_i g_j^{-1}|$, then $\bar{f} = (f_1, \cdots, f_{n^2})$ is identically zero precisely when the three conditions of the lemma are met. One solution to this system occurs when $\bar{X}$ has 1's in positions corresponding to x_{ii}'s and zeros elsewhere (that is, the X_i are the standard basis points) and $\bar{\delta} = (\sqrt{2}, \ldots, \sqrt{2})$. Call this point (A, B).

By the Implicit Function Theorem, if the matrix we get by taking the partial derivatives of the f_t's with respect to the x_{ij}'s and evaluating at the point (A, B) is invertible, then there is a neighborhood U of B and a neighborhood V of (A, B) so that for each $\bar{\delta} \in U$ there is a unique point $\bar{X}$ so that $(\bar{X}, \bar{\delta})$ in V and $\bar{f}(\bar{X}, \bar{\delta}) = \bar{0}$. Since U is open, we can choose $|g_2|, \ldots, |g_n|$ so that $|g_j| = |g_i|$ if and only if $g_j = g_i$ or $g_j = g_i^{-1}$, $\delta_{ij} = |g_i g_j^{-1}|$ and so that $\bar{\delta}$ is in U.

Now we need to show that the matrix of partial derivatives of the f_t's with respect to the x_{ij}'s evaluated at (A, B) is invertible. The order of the f_t's is helpful: in the matrix of partial derivatives the k^{th} row contains the partial derivatives of f_k with respect to the various x_{ij}'s, and the ℓ^{th} column contains the partial derivatives of the various f_t's with respect to x_{ij} where $\ell = (i-1)n + j$.

Case 1. If $s < r$, we have that $f_t(\bar{X}, \bar{\delta}) = x_{sr}$ so $\frac{\partial f_t}{\partial x_{ij}} = \begin{cases} 0 & \text{if } (i,j) \neq (s,r) \\ 1 & \text{if } (i,j) = (s,r) \end{cases}$.

Thus we get non-zero entries only in the diagonal positions of the rows where $s < r$.

Case 2. If $s = r$, $f_t(\bar{X}, \bar{\delta}) = x_{s1}^2 + \cdots + x_{sn}^2 - 1$ and $\frac{\partial f_t}{\partial x_{ij}} = \begin{cases} 0 & \text{if } i \neq s \\ 2x_{sj} & \text{if } i = s \end{cases}$. Further since the only x_{sj} that is non-zero is x_{ss}, after evaluating this becomes $\frac{\partial f_t}{\partial x_{ij}} = \begin{cases} 2 & \text{if } i = j = s \\ 0 & \text{otherwise} \end{cases}$. Thus we get non-zero entries only in the diagonal positions of the rows where $s = r$.

Case 3. If $s > r$, $f_t(\bar{X}, \bar{\delta}) = (x_{s1} - x_{r1})^2 + \cdots + (x_{sn} - x_{rn})^2 - \delta_{sr}^2$. Then $\frac{\partial f_t}{\partial x_{ij}} = \begin{cases} 0 & \text{if } i \neq r, s \\ 2(x_{sj} - x_{rj}) & \text{if } i = s \\ -2(x_{sj} - x_{rj}) & \text{if } i = r \end{cases}$. After evaluation this becomes

$$\begin{cases} 2x_{ss} & \text{if } i = j = s \text{ or } i = r, j = s \\ -2x_{rr} & \text{if } i = j = r \text{ or } i = s, j = r \\ 0 & \text{otherwise} \end{cases}.$$

Thus the rows where $s > r$ have non-zero entries in columns $(s-1)n + s, (r-1)n + r, (r-1)n + s$ and $(s-1)n + r$. The last of these is a non-zero entry in the diagonal position of this row. The first two non-zero entries can be eliminated using the single non-zero entry in each of rows $(s-1)n + s$ and $(r-1)n + r$ (Case 1). Further the non-zero entry in column $(r-1)n + s$ can be eliminated using the single non-zero in the diagonal position of row $(r-1)n + s$ (Case 2).

Thus using a little Gaussian elimination we can transform the matrix of partial derivatives into a full rank diagonal matrix. Thus the matrix of partial derivatives is invertible and we are guaranteed the existence of the lengths of the $|g_i|$ and the points X_i that we desire.

Each X_i is a small perturbation from the i^{th} vector in the standard basis. It is immediate that the X_i's form a basis. □

In the proof above the position vectors of the n group points form a basis in $\mathbb{R}^n$. If we translate W so that one of the group points is located at the origin, then the new position vectors of the other n-1 group points are linearly independent and span an (n-1)-dimensional subspace of $\mathbb{R}^n$. The orientation points and midpoints are also contained in this subspace, and the action of G on the translated point set extends to an action by isometries on the subspace. Thus:

Corollary 1.1. If G is a group of order n, then there exists $W \subset \mathbb{R}^{n-1}$ so that the isometry group of W is isomorphic to G, and the action of G on W extends to an action by isometries on $\mathbb{R}^{n-1}$.

Corollary 1.2. For every subgroup $H < G$, there is a modification of the points of W so that H is isomorphic to the full isometry group of the modified point set.

Proof. We can "construct" a set W for H in virtually the same way as we did for G. The only difference is that the H-edge orbits are more numerous, and smaller, than the G-edge orbits. That is, a single G-edge orbit may split into multiple H-edge orbits, each with a different assigned edge length. Using these edge orbit lengths, rather than distances assigned to group elements, our Lemma 1 follows with no difficulty. Thus there exists a set n points in $\mathbb{R}^n$, each labeled by an element of G on which H acts by isometries. We add in $\binom{n}{2}$ orientation points and midpoints as appropriate to H. To see that the isometry group of these points is exactly H, we use the facts that edges between group points have equal length if and only if they are in the same H-edge orbit and that the H-stabilizer of every group point is trivial. The proof for H is the same as the proof for G. $\square$

3 The Isometry Dimension of Small Groups

We know that we can realize a finite group G by a set of points in $\mathbb{R}^{|G|-1}$, but what is the smallest dimension in which G can be realized? We call this the *isometry dimension* of G and denote it by $\delta(G)$.

An isometry of a subset of $\mathbb{R}$ can be extended to a reflection or a translation of $\mathbb{R}$. Since translations have infinite order and the product of two reflections is a translation, $\mathbb{Z}_2$ is the only finite group with isometry dimension 1. As mentioned in the introduction, if we take the vertices of a regular n-gon in $\mathbb{R}^2$ we realize the dihedral group of order $2n$. This same set with orientation points realizes $\mathbb{Z}_n$. The only group of order less than eight that is neither cyclic nor dihedral is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This can be realized in $\mathbb{R}^2$ by the vertices of a rectangle. Thus groups of order three through seven have isometry dimension two.

The groups of order eight show more contrast. We know that $\delta(\mathbb{Z}_8) = \delta(D_4) = 2$. We show below that $\delta(\mathbb{Z}_4 \times \mathbb{Z}_2) = 3$. A similar argument shows that $\delta(\mathbb{Z}_2^3) = 3$. Finally we establish that the isometry dimension of the quaternions is at least 4.

Theorem 2. $\delta(\mathbb{Z}_4 \times \mathbb{Z}_2) = 3$.

Proof. Take the vertices of a square in the xy-plane centered about the origin and insert orientation points so that its isometry group is $\mathbb{Z}_4$. Let $W \subset \mathbb{R}^3$ consist of these eight points together with the points $\{(0,0,1),(0,0,-1)\}$. The isometry group of W is $\mathbb{Z}_4 \times \mathbb{Z}_2$. Thus $\delta(\mathbb{Z}_4 \times \mathbb{Z}_2) \leq 3$.

Now suppose $W \subset \mathbb{R}^2$ realizes $\mathbb{Z}_4 \times \mathbb{Z}_2$. Let α be a generator of order four and β a generator of order two. There must exist $x \in W$ that is moved by α^2. Thus $\{x, x\alpha, x\alpha^2, x\alpha^3\}$ forms a quadrilateral. Since α and α^2 are isometries, this quadrilateral has equal length sides and equal length diagonals. Thus the α-orbit of x is a square. If $\beta(x) = \alpha^i(x)$, then $\alpha^{-i}\beta$ is a group generator of order two that fixes x, so we might as well assume β fixes x. If $\beta(x) = x$, since β and α commute, it fixes the α-orbit of x and so there must exist y that is moved by β. Since the entire α-orbit of x must lie on the perpendicular bisector of the segment from y to $y\beta$, W cannot be a subset of $\mathbb{R}^2$. Finally if β moves x outside its α-orbit, consider $\{x, x\alpha\beta, x\alpha^2, x\alpha^3\beta\}$, the points on the $\alpha\beta$-orbit of x. As before these must form a square, and this square shares exactly one diagonal with $\{x, x\alpha, x\alpha^2, x\alpha^3\}$. This cannot occur in $\mathbb{R}^2$. $\square$

Let $\mathcal{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the quaternions.

Theorem 3. $\delta(\mathcal{Q}) \geq 4$.

Proof. Suppose $W \subset \mathbb{R}^n$ realizes $\mathcal{Q}$. There exists $x \in W$ that is moved by -1. If x_γ denotes the image of the point x under the group operation γ, then $\{x_{\pm 1}, x_{\pm i}, x_{\pm j}, x_{\pm k}\} \subset W$ is a set of eight distinct points. Now $dist(x_1, x_i) = dist(x_{-1}, x_{-i}) = dist(x_i, x_{-1}) = dist(x_{-i}, x_1)$ since these distances are the edge lengths of images of $[x_1, x_i]$ under the isometries determined by $1, -1, i$, and $-i$. Thus x_i and x_{-i} both lie on the perpendicular bisector of the segment from x_1 to x_{-1} as do $x_{\pm j}$ and $x_{\pm k}$. These six points are in a hyperplane perpendicular to $x_1 - x_{-1}$. Similarly the four distances $dist(x_{\pm i}, x_{\pm j})$ are all equal and thus within this hyperplane there are four points $x_{\pm j}, x_{\pm k}$ all on the subhyperplane perpendicular to $x_i - x_{-i}$. Continuing in the same fashion we find two points $x_{\pm k}$ on the sub-subhyperplane perpendicular to $x_j - x_{-j}$. For these two points to be distinct it is necessary that $n \geq 4$.

$\square$

4 Questions

The following arise naturally:

Question 1. Is $\delta(\mathcal{Q}) = 4$?

Question 2. Is there a relation between the order of a group and the minimum number of points needed to realize it?

Conjecture 1. It is easy to see that $\mathbb{Z}_2^n$ is realized by the vertices of an n-dimensional (non-regular) octahedron. Thus $\delta(\mathbb{Z}_2^n) \leq n$. We conjecture that equality holds.

Question 3. We know that $\mathbb{Z}_2$ is the only finite group with isometry dimension one. Which finite groups have isometry dimension two? (three??)

Question 4. We know that $\delta(G) \leq |G| - 1$. Equality holds here if G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Does equality hold anywhere else? for large groups? Does there exist a constant c such that $\delta(G) < c \cdot log(|G|)$?

Question 5. S_n acts by permutations on the vertices of an n-simplex in $\mathbb{R}^{n-1}$. Is $\delta(S_n) = n - 1$?

Question 6. If $H < G$ is $\delta(H) \leq \delta(G)$?. How about if G is S_n?

Question 7. Is $\delta(G \times G) = 2\delta(G)$? Is $\delta(G \times H) = \delta(G) + \delta(H)$?

Question 8. The set $W = \mathbb{Z} \cup \{n + \frac{\sqrt{2}}{2} : n \in \mathbb{Z}\}$ realizes $\mathbb{Z}$. What infinite groups can be realized by subsets of Euclidean space?

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