Thickness-Two Graphs Part One:  
New Nine-Critical Graphs, Permutated Layer Graphs, and Catlin’s Graphs  

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Abstract  

The purpose of this paper is to offer new insight and tools toward the pursuit of the largest chromatic number in the class of thickness-two graphs. At present, the highest chromatic number known for a thickness-two graph is 9, and there is only one known color-critical
such graph. We introduce 40 small 9-critical thickness-two graphs, and then use a new construction, the permuted layer graphs, together with a construction of Hajós to create an infinite family of 9-critical thickness-two graphs. Finally, a non-trivial infinite subfamily of Catlin’s graphs, with directly computable chromatic numbers, is shown to have thickness two.

1 Introduction

Decomposing a graph into planar layers is of primary importance in the field of VLSI (see, for example, [GJS76]). To this end, the thickness of a graph $G$, denoted $\Theta(G)$, is the smallest integer $t$ so that $G$ can be represented as the union of $t$ planar graphs. We then say that $G$ is thickness $t$ [Wes01]. Thus planar graphs have thickness one, while $K_5$ and $K_{3,3}$ have thickness two. The chromatic number of $G$, denoted $\chi(G)$, is the smallest integer $k$ such that $G$ can be properly vertex colored with $k$ colors. We then say that $G$ is $k$-chromatic. Thus, for example, if $G$ is planar then $\chi(G) \leq 4$ [AH76, RSST96]. A graph is said to be $k$-critical if it is $k$-chromatic but every proper subgraph can be properly colored with fewer than $k$ colors.

Given an arbitrary graph $G$ and fixed positive integer $t > 1$, verifying that $\Theta(G) = t$ is an NP-hard problem [Man83]. Similarly, given an arbitrary graph $G$ and fixed positive integer $k > 2$, verifying that $\chi(G) = k$ is also NP-hard [GJ90, GJS74]. Combining these two ideas leads to a longstanding and difficult open problem [Rin59]: “What is the largest chromatic number that occurs in the class of thickness-two graphs?”

It is well-known that the largest chromatic number of a thickness-two graph lies between 9 and 12 inclusive [JT95, Hut93]. That is, if $G$ is an arbitrary thickness-two graph then $\chi(G) \leq N$, where

$$N \in \{9, 10, 11, 12\}. \quad (1)$$

The largest element in (1) comes from a straightforward induction argument that relies on Euler’s Formula for plane graphs. The smallest element in (1) is due to the existence of the 9-critical thickness-two graph given by $S = K_6 \lor C_5$, which is called Sulanke’s graph. The third author proved that $S$ had thickness two in 1973 and the result was reported in 1980 in [Gar80]. Since then no progress has been made toward closing this gap of possibilities. Further, no other examples of 9-critical thickness-two graphs...
have emerged. The desideratum for the main question would be to find an asymptotically sharp bound for $N$ in (1). Short of that difficult goal, producing a 10-chromatic thickness-two graph would be of paramount importance. In order to construct the latter, new techniques are needed, as well as a large collection of 9-critical thickness-two graphs. It is the latter upon which we focus our attention in this paper and the sequel [GS].

In particular, the purpose of this paper is threefold. First a new class of thickness-two graphs, called permuted layer graphs, is introduced and shown to contain all thickness-two graphs as subgraphs. Thus we can study permuted layer graphs to obtain the chromatic properties of thickness-two graphs. Second, we introduce infinitely many 9-critical thickness-two graphs and we do so in two stages. In particular we give a list of 40 new 9-critical thickness-two graphs on 12 through 16 vertices. Moreover, to add further ballast to the lower bound of 9 in (1), we use a construction of Hajós together with the Permuted Layer construction to generate infinitely many 9-critical thickness-two graphs. To strengthen the latter, we characterize precisely when the construction of Hajós produces a graph that has thickness $t$. Third, an infinite subfamily of Catlin’s graphs [Cat79] with easily computable chromatic numbers is shown to have thickness two. We end with a list of open questions.

2 Permuted Layer Graphs

In this section we define a restricted type of thickness-two graph, called a permuted layer graph, and identify two especially nice properties of the smaller class. First, permuted layer graphs are easy to construct. Second, every maximal thickness-two graph is a permuted layer graph. Thus there is a thickness-two graph with chromatic number as large as $k$ if and only if there is a permuted layer graph with chromatic number as large as $k$. The following construction defines the new subclass of graphs.

The Permuted Layer Construction. Let $H$ be a planar graph with $V(H) = \{v_1, \ldots, v_n\}$, and let $\sigma$ be a permutation of $V(H)$. Construct another planar graph $H'$ isomorphic to $H$ with vertices labeled so that the vertex corresponding to $v_i$ in $H$ is labeled $\sigma(v_i)$ in $H'$. Identify $H$ and $H'$ at vertices of the same label and call the resulting graph $\tilde{G}$; that is $\tilde{G} = H \cup H'$. Since $\tilde{G}$ may have multiple edges, we let $G$ be the underlying simple graph and call
it a \textit{permuted layer graph with base graph} \( H \). In the special situation when \( \tilde{G} \) does not have multiple edges we call \( G (= \tilde{G}) \) a \textit{full permuted layer graph}.

A more direct, but slightly less intuitive, way to obtain the permuted layer graph from the underlying graph \( H \) and the vertex permutation \( \sigma \) is to add the edge set \( \{ (\sigma(a), \sigma(b)) \mid (a, b) \in E(H) \} \) to \( H \) and then remove multiple edges. For example, Figure 1 shows a representation of \( K_6 \) as a permuted layer graph.

![Figure 1: A permuted layer representation of \( K_6 \) using vertex label permutation \((12)(34)(56)\). Edges (1,2), (3,4), and (5,6) are removed from the second layer.](image)

Note that if \( \tau \in \text{Aut}(H) \) and \( \sigma \in \text{Perm}(V(H)) \) then \( \sigma \) and \( \sigma \tau \) produce isomorphic permuted layer graphs. Thus, the number of distinct permuted layer graphs with base \( H \) is at most \( \frac{|V(H)|!}{|\text{Aut}(H)|} \). However, if we restrict our analysis to full permuted layer graphs, the size of the set can be much smaller, as evidenced by the next proposition.

**Proposition 1.** \textit{There is a unique full permuted layer graph whose base graph is the icosahedral graph.}

\textit{Proof.} The labeling of the vertices of the icosahedral graph is given by the lefthand graph in Figure 2. The permutation of vertex labels given by \( \sigma = (2 10 3 7)(4 9 6 8)(5 11) \) yields a second layer that does not produce any multiple edges, as is shown in the righthand graph Figure 2. Thus there is at least one full permuted layer graph whose base graph is the icosahedral graph.
Figure 2: The unique full permuted layer graph whose base graph is the icosahedral graph.

On the other hand, let $G$ be a full permuted layer graph whose base graph is the icosahedral graph $H$. Let $\sigma$ be a vertex permutation that produces $G$ from $H$. Since $G$ is a full permuted layer graph the edge set $\{(\sigma(a), \sigma(b)) \mid (a, b) \in E(H)\}$ is disjoint from $\{(a, b) \in E(H)\}$. Thus every vertex in $G$ has twice the degree it had in the icosahedral graph; that is, $G$ has 12 vertices each of degree 10. Then $G^c$ is a matching. Since all matchings on 12 vertices are isomorphic, all such $G$ are isomorphic, and thus $G$ is unique.

More generally, the class of permuted layer graphs contains all of the thickness-two graphs as subgraphs.

**Proposition 2.** Every thickness-two graph is a subgraph of a permuted layer graph.

**Proof.** Let $H$ be a thickness-two graph with (edge-disjoint) planar layers $H_1$ and $H_2$. Suppose $|V(H)| = n$. Create a permuted layer graph, $G$, as follows:

Procedure **Permuted_Layer**($H = H_1 \cup H_2$)

1. Index the vertices of $H$ (and hence of $H_1$ and $H_2$) by $v_1, v_2, \ldots, v_n$. Now re-index the vertices of $H_2$ by $v_{n+1}, \ldots, v_{2n-1}, v_n$ in order.
2. Identify $H_1$ and $H_2$ at $v_n$ yielding a planar graph $G_a$. In $G_a$ the vertices $v_1, \ldots, v_n$ induce $H_1$ and the vertices $v_n, \ldots, v_{2n-1}$ induce $H_2$.
3. Let $\sigma = (v_1 v_{n+1})(v_2 v_{n+1}) \ldots (v_{n-1} v_{2n-1})$. Note that $\sigma$ is a product of
transpositions and that it fixes $v_n$.

4. Let $G$ be the permuted layer graph obtained from the base $G_a$ using the vertex permutation $\sigma$.

Note that the second layer of $G$ is isomorphic to $G_a$ but the vertex labels have been changed so that $v_1, \ldots, v_n$ induce $H_2$ while $v_n, \ldots, v_{2n-1}$ induce $H_1$. Therefore in $G$ the vertices $v_1, \ldots, v_n$ induce both layers of $H$ and thus $H$ itself (as do the vertices $v_n, \ldots, v_{2n-1}$). Thus $G$ is a permuted layer graph containing $H$ as a subgraph.

For example, the union of the two planar graphs in Figure 11 in Section 3.2, including edges (4,11) and (11, 12) (dashed) and excluding edge (4, 12) (bold), is a full permuted layer graph that contains Sulanke’s graph as a subgraph.

3 New 9-critical Thickness-Two Graphs

To date there is only one published 9-critical thickness-two graph: Sulanke’s graph $S$, due to the third author, is given by $S = K_6 \lor C_5$. An edge-disjoint decomposition of $S$ as a permuted layer graph is shown in Figure 3. The $K_6$ is induced by vertices 1-6, and the $C_5$ is induced by vertices 7-11. In the next subsection, we introduce 40 new 9-critical thickness-two graphs.

![Figure 3: $K_6 \lor C_5$ thickness-two decomposition as a permuted layer graph.](image)

3.1 Forty New Small 9-critical Thickness-Two Graphs

In this subsection we introduce 40 new 9-critical thickness-two graphs on 12 through 16 vertices. The graphs were selected from the hundreds found
because they exhibit a wide range of values for the number of vertices and edges. Each graph \( G \) will be identified by its degree sequence and a reference number. The reference number is formatted as \( v.e.n \), where \( v = |V(G)| \), \( e = |E(G)| \), and \( n \) is a counter indicating that \( G \) is the \( n \)th 9-critical graph on \( v \) vertices and \( e \) edges in our collection. For those subsets of graphs on the same number of vertices and edges (i.e., for which \( n > 1 \)) we offer only such ones that have a unique degree sequence, which ensures that all graphs presented are distinct. Not all of the examples require a degree sequence for verification of uniqueness (those for which \( n = 1 \)) but the labeling scheme is crafted with continued work and expansion in mind. Thus we give full information for each graph listed here.

The generation of 9-critical graphs that have few enough edges to be thickness-two was accomplished with a modified version of \textit{geng}, which is a component of Brendan McKay’s graph isomorphism software \textit{nauty} [McK81, McK07a]. The thickness-two decompositions were found by randomly flipping diagonals of planar triangulations [Sul]. The pruning of those 9-chromatic graphs for which a thickness-two decomposition was found and that were not initially 9-critical was done with \textit{nauty} as well. That each graph has thickness two was then verified with \textit{Groups and Graphs} [KK05, McK07b] in order to display a thickness-two embedding for each example. The final graphics were generated by \textit{Mathematica} [Wol07]. The 40 new 9-critical thickness-two graphs follow.

![Figure 4: New 9-critical thickness-two graphs.](image_url)
Figure 5: New 9-critical thickness-two graphs, continued.
Figure 6: New 9-critical thickness-two graphs, continued.
Figure 7: New 9-critical thickness-two graphs, continued.
Figure 8: New 9-critical thickness-two graphs, continued.
Figure 9: New 9-critical thickness-two graphs, continued.
Figure 10: New 9-critical thickness-two graphs, continued.
3.2 Infinitely Many 9-critical Thickness-Two Graphs

In this section we provide a technique for constructing infinite families of \( k \)-critical thickness-two graphs. The construction requires two tools, one of which is the Procedure **Permuted Layer** of Section 2, which is used in tandem with the well-known *Hajós construction* [Haj61, JR99, Urq97, Wes01, Wes83]. In addition, two (not necessarily distinct) \( k \)-critical thickness-two graphs are required to initialize the procedure. The Hajós construction is described next.

**The Hajós Construction.**

**Step One:** Let \( G_1 \) and \( G_2 \) be two \( k \)-critical graphs that are identified at a vertex \( v \). If \((v, w_1) \in E(G_1)\) and \((v, w_2) \in E(G_2)\) then the graph \( G = (G_1 - (v, w_1)) \cup (G_2 - (v, w_2)) \cup (w_1, w_2)\) is \( k \)-critical.

**Step Two:** Identification of any independent set of vertices in \( G \) yields a graph \( \hat{G} \) such that \( \chi(\hat{G}) \geq k \). The graph \( \hat{G} \), however, is not necessarily \( k \)-critical.

**Proposition 3.** Let \( H \) and \( K \) be graphs. Let \( G \) be the Step-One-Hajós of \( H \) and \( K \). Then \( \Theta(G) = \max\{\Theta(H), \Theta(K)\} \).

**Proof.** Since \( G \) is the Step-One-Hajós of \( H \) and \( K \), there exist vertices \( u, v, x \) so that \( x \) is the vertex at which \( H \) and \( K \) are identified in \( G \), \( u \in V(H), v \in V(K) \), \( ux \in E(H), (v, x) \in E(K) \) in which case \( G = H \cup K \cup \{(u, v)\} - \{(u, x), (v, x)\} \).

Suppose that \( \Theta(G) = t \). Partition \( G \) into edge-disjoint planar layers \( L_1, \ldots, L_t \) and assume without loss of generality that \((u, v)\) is an edge of \( L_1 \). Since \( L_1, \ldots, L_t \) are planar and \( x \) is a cut vertex of \( G - \{(u, v)\} \), there are planar graphs \( H_1, \ldots, H_t \) and \( K_1, \ldots, K_t \) so that \( H_1 \) and \( K_1 \) identified at \( x \) yield \( L_1 - \{(u, v)\} \), and for all \( i > 1 \), \( H_i \) and \( K_i \) identified at \( x \) yield \( L_i \).

Notice that \( H = (H_1 \cup \{(u, x)\}) \cup H_2 \cup \ldots \cup H_t \) and \( K = (K_1 \cup \{v, x\}) \cup K_2 \cup \ldots \cup K_t \). Further, if \( H_1 \cup \{(u, x)\} \) is planar then there is a decomposition of \( H \) into \( t \) planar layers. Similarly, if \( K_1 \cup \{(v, x)\} \) is planar, there is a decomposition of \( K \) into \( t \) planar layers. In particular, if both \( H_1 \cup \{(u, x)\} \) and \( K_1 \cup \{(v, x)\} \) are planar then \( \max\{\Theta(H), \Theta(K)\} \leq \Theta(G) \).

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Suppose to the contrary that $H_1 \cup \{(u,x)\}$ is not planar. Then by Kuratowski’s Theorem there exists a subgraph $S$ of $H_1 \cup \{(u,x)\}$ that is a subdivision of $K_5$ or $K_{3,3}$. Since $H_1$ is planar, any such subdivision must include $(u,x)$ as an edge.

Note that if $K_1$ is not connected we may add edges to make it connected while maintaining the planarity of $L_1$. Thus we may assume without loss of generality that $K_1$ is connected in which case there is a path $P$ in $K_1$ from $u$ to $x$. By replacing $(u,x)$ by $(u,v)P$ in $S$ we produce a subdivision of $K_5$ or $K_{3,3}$ in $L_1$. That is, $S' = S \cup P \cup \{(u,v)\} - \{(u,x)\}$ is a subdivision of $K_5$ or $K_{3,3}$ that is a subgraph of $L_1$, which is a contradiction since $L_1$ is planar.

It remains to show that $\max\{\Theta(H), \Theta(K)\} \geq \Theta(G)$. Suppose to the contrary that $\max\{\Theta(H), \Theta(K)\} = s < t$. Then each of $H$ and $K$ decompose into edge-disjoint planar layers $H_1, \ldots, H_s$ and $K_1, \ldots, K_s$ (note that some may be trivial since $s$ is the maximum of two thicknesses). Define $L_1 = H_1 \cup K_1 \cup \{(u,v)\} - \{(u,x)\}$. For $i = 2, \ldots, s$, define $L_i = H_i \cup K_i$. Clearly for $i > 1$, $L_i$ is planar.

Since $H_1$ is planar and $(u,x)$ is an edge of $H_1$ there is a plane drawing of $H_1$ with $ux$ on the infinite face. Similarly there is a plane drawing of $K_1$ with $(v,x)$ on the infinite face. Utilizing these particular drawings there is a plane drawing of $H_1$ and $K_1$ merged at vertex $x$ with both $(u,x)$ and $(v,x)$ on the infinite face. Removing $(u,x)$ and $(v,x)$ and then adding $(u,v)$ can be accomplished without any edge crossings. Hence $L_1$ is planar.

In that case $G = L_1 \cup \ldots \cup L_s$ is a decomposition of $G$ into $s < t$ planar layers, which is impossible since $\Theta(G) = t$. Thus $\Theta(G) \geq \max\{\Theta(H), \Theta(K)\}$. Altogether $\max\{\Theta(H), \Theta(K)\} = \Theta(G)$ as desired. $\square$

The characterization given in Proposition 3 provides a means to a recursive algorithm to construct infinitely many 9-critical thickness-two graphs: all that is required is either one or two 9-critical thickness-two graph as the initial input.

**Corollary 1.** There are infinitely many 9-critical thickness-two graphs.

Penultimately for this section, we note that the pair of graphs in Figure 11 with the dashed edges (4,11) and (4,12) removed and the bold edge (4,12) included is an example of the Step-One-Hajós of two copies of the Sulanke graph.
Finally we note that the procedure outlined by the proof of Proposition 3 implies that if there is one example of a $k$-critical thickness-two graph, then there exist infinitely many $k$-critical thickness-two graphs, thus providing asymptotically good bounds for possible improvements to (1) in Section 1.

\section{Catlin’s Graphs}

Typically, when one looks for high chromatic thickness-two graphs, a candidate graph is presented for which either the chromatic number or thickness,
but not both, is known. Paul Catlin [Cat79] identified a family of graphs $C_n[K_r]$ in which each vertex of an $n$-cycle is expanded to a $K_r$ and the join of “adjacent” copies of $K_r$ is taken. Catlin’s motivation was to show that there exist $m$-chromatic graphs that contain no subdivision of $K_n$. Toward this end he provides a formula for the chromatic number of $C_n[K_r]$. In this section, we show that every Catlin graph $C_n[K_3]$ with $n \geq 4$ has thickness two. This yields an infinite family of non-trivial graphs for which both the chromatic number and thickness are known. In particular, it is an easy consequence of [Cat79] that

$$\chi(C_n[K_3]) = \begin{cases} 8 & \text{if } n = 5 \\ 7 & \text{if } n \text{ is odd and } n \geq 7 \\ 6 & \text{if } n \text{ is even and } n \geq 4. \end{cases}$$

The thickness-two decomposition of $C_n[K_3]$ is described next.

**Decomposition of $C_n[K_3]$ into two plane layers.** In what follows, $P_m$ is a simple path with $m$ vertices. Let $G_n = C_n[K_3]$, and label the vertices of $C_n$ by $v_1, v_2, \ldots, v_n$. We expand vertex $v_i$ to a $K_3$ with vertices labeled by \{3i−2, 3i−1, 3i\} for $i = 1, \ldots, n$. The edges of $C_n[K_3]$, in addition to those of each of the $n$ copies of $K_3$, are given by the join of neighboring copies of $K_3$. The first layer of the thickness-two decomposition of $G_n$, written $G_{n,1}$, is given by the edge-disjoint union of three copies of $P_n$, $n$ 3-cycles, and a $P_{3n−2}$. Specifically, the three $P_n$s are \{1, 4, 7, \ldots, 3n−5, 3n−2\}, \{n−1, 2, 5, \ldots, 3n−7, 3n−4\}, and \{3, 6, \ldots, 3n−3, 3n\}; the $n$ 3-cycles are the subgraphs induced by vertices \{3i+1, 3i−1 (mod 3n), 3i + 3\}$_{i=0}^{n−1}$, the $P_{3n−2}$ is given by \{3, 2, 1, 6, 5, 4, \ldots, 3i, 3i−1, 3i−2, \ldots, 3n−3, 3n−4, 3n−5, 3n\}. Then $G_{n,1}$ is embedded in the plane by drawing the 3-cycles as concentric congruent triangles, and arranging the vertices so that the three $P_n$s are each on a line. The remaining edges are the ones belonging to the $P_{3n−2}$, each one of which is the diagonal of a face of size four; see the left graph in Fig. 2 for a plane drawing of $G_{5,1}$. Note that $|E(G_{n,1})| = 9n−6$ and so $G_{n,1}$ is edge maximal since $|V(G_{n,1})| = 3n$.

Finally, the complement of $G_{n,1}$ in $G_n$, denoted $G_{n,2}$, is given by the edge-disjoint union of a $P_{3n−4}$ and another planar graph $G'$ with eight vertices and 11 edges. The latter can be embedded inside a plane drawing of the cycle induced by the $P_{3n−4}$. The order of the vertices in the path is given by 3, 4, 8, 6, 7, 11, 9, 10, \ldots, 3i−1, 3i−3, 3i−2, \ldots, 3n−1, 3n−3, 3n−2. The
remaining vertices of $G_{n,2}$ are 1, 2, 5, and $3n$; the subgraph induced by these vertices together with vertices 3, $3n-2$, $3n-1$, and $3n-4$, is shown in dashed edges in the right graph in Figure 12. Note that $|E(G_n)| = 12n$, $|E(G_{n,1})| = 9n - 6$, and $|E(G_{n,2})| = 3n + 6$. Thus $|E(G_n)| = |E(G_{n,1})| + |E(G_{n,2})|$, as expected.

5 Open Questions

1. Figure 3 is a representation of Sulanke’s graph as a permuted layer graph. We have shown by brute force that Sulanke’s graph is not a full permuted layer graph; find a direct proof that this is so.

2. The permuted layer construction, as described in Section 2, requires as input a planar graph $H$ on $n$ vertices and a permutation of vertex labels $\sigma \in S_n$. The construction yields a thickness-at-most-two graph, which is the simple graph underlying $H \cup \sigma(H)$. Define $\beta(H)$ to be the number of distinct full permuted layer graphs generated by base planar graph $H$.

   (a) For which (planar) $H$ is $\beta(H) = 1$? We know, for example, that if $H$ is the icosahedral graph, then $\beta(H) = 1$; see Proposition 1.

   (b) What can be said of the following decision problem: given a positive integer $k$, is $\beta(H) = k$?

   (c) If $\sigma \in Aut(H)$ then $H \cup \sigma(H)$ is clearly planar. If $H$ is maximal planar then $H \cup \sigma(H)$ is planar only if $\sigma \in Aut(H)$. For which planar graphs $H$ and which $\sigma \notin Aut(H)$ is $H \cup \sigma(H)$ planar?

3. Are any of the 9-critical graphs in Section 3.1 (full) permuted layer graphs?

4. Step 2 of the Hajós construction identifies nonadjacent vertices. Doing so either preserves or increases the chromatic number. Find a criteria under which nonadjacent vertices can be identified while preserving or lowering the thickness.

5. For which $n \geq 4$, if any, is $C_n[K_3]$ a (full) permuted layer graph?

6. Find a 10, 11, or 12-chromatic thickness-two graph, or prove that none exists.
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References


