The Thickness and Chromatic Number of r-Inflated Graphs

Michael O. Albertson
L. Clark Seelye Professor
Department of Mathematics and Statistics
Smith College, Northampton MA 01063

Debra L. Boutin
Department of Mathematics
Hamilton College, Clinton, NY 13323
dboutin@hamilton.edu

Ellen Gethner
Department of Computer Science
Department of Mathematics
University of Colorado Denver, Denver, CO 80217
ellen.gethner@ucdenver.edu

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Dedicated to Carsten Thomassen on the occasion of his 60th birthday.

Abstract

A graph has thickness t if the edges can be decomposed into t and no fewer planar layers. We study one aspect of a generalization of Ringel's famous Earth-Moon problem: what is the largest chromatic number of any thickness-2 graph? In particular, given a graph G we consider the r-inflation of G and find bounds on both the thickness and the chromatic number of the inflated graphs. In some instances the best possible bounds on both the chromatic number and thickness are achieved. We end with several open problems.

1 Introduction

The Four Color Problem [Ore67], now Theorem [AH77, AHK77, AH76b, AH76a, RSST96], has inspired a rich body of literature from which many open questions remain. We study a generalization of a problem posed by Gerhard Ringel in 1959 [Rin59] (see also [Hut93]) that itself is a natural generalization of the Four Color Theorem: *what is the largest chromatic number of any thickness-2 graph*? More generally,

What is the largest chromatic number of any thickness-t graph?

The thickness of a graph is one possible measurement of "closeness to planarity" [HR03]. In particular, a graph G is said to have *thickness t*, written $\Theta(G) = t$, if the edges of G can be partitioned into t sets each of which induces a planar graph, and t is smallest possible. The general question has been answered definitively only in the case when $t = 1$, which is, of course, due to the The Four Color Theorem. Let f_t be the largest chromatic number of any thickness-t graph. The most general statement that can be made is [JT95]:

$$f_t \in \begin{cases} \{4\} & \text{if } \; t = 1 \\ \{9, 10, 11, 12\} & \text{if } \; t = 2 \\ \{6t-2, 6t-1, 6t\} & \text{if } \; t > 2 \end{cases} . \tag{1}$$

The Sulanke graph (due to Thom Sulanke, reported in [Gar80]) was the only 9-critical thickness-two graph that was known from 1973 through 2007. In 2008, the Sulanke graph was used to construct an infinite family of 9-critical thickness-two graphs and hence for $t = 2$ any improvement in (1) will be asymptotic [BGS08]. The same technique can be easily adapted to show that if one k-critical thickness-t graph exists, then there are infinitely many such graphs. Thus any improvement to (1) for any value of t will be asymptotic as well.

In [BGS08], the second and third authors of this paper studied a particular family of Catlin's graphs. The historical motivation for these graphs was to disprove a conjecture of Hajós that an s-chromatic graph necessarily contains a subdivision of K_s. Catlin's graphs can be described in a variety of ways, one of which is as the lexicographic product of cycle C_n with the complete graph K_r. Another is the r-uniform replication of C_n [Tho05], and the final one (which we use) is the r-inflation of C_n [PST03].

In this article we expand on the idea of Catlin's graphs by considering the 2-inflation and then the r-inflations of a variety of graphs; we write $G[K_r]$ or $G[r]$ to indicate the r-inflation of graph G. We begin investigations of both the thickness and chromatic numbers of graphs inflated in this way. In this first article, the major emphasis is on the thickness with some naturally following results on chromatic number. In a subsequent article, the emphasis will be on the chromatic number of r-inflated graphs.

2 Terminology and Observations

Definition 1. Let G be a graph and define the *r-inflation* of G to be the lexicographic product $G[K_r]$, denoted by $G[r]$. In the special case that $r = 2$, we call $G[2]$ the *clone* of G.

Recall that the lexicographic product $G[H]$ replaces every vertex of G with a copy of H and places edges between all pairs of vertices in copies of H that are associated with adjacent pairs of vertices of G. That is, the vertex set of $G[H]$ is $V(G) \times V(H)$ and there is an edge between (g_1, h_1) and (g_2, h_2) if and only if $g_1 = g_2$ and h_1 is adjacent to h_2 in H (the vertices are in the same copy of H) or g_1 is adjacent to g_2 in G (the vertices are in copies of H associated with adjacent vertices in G). See, for example, [Wes01].

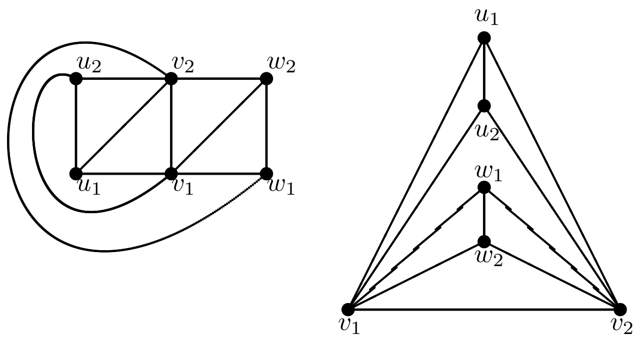

Figure 1: Two plane drawings of the clone of P_3.

From another point of view, we obtain $G[r]$ by replacing each vertex of G by K_r and replacing each edge of G by $K_{r,r}$ (the join of the neighboring K_rs). See, for example, [PST03]. An r-inflation of G has the following properties, all of which are straightforward to verify.

Observations:

1. If the number of vertices and edges of G are V and E, then the number of vertices and edges of $G[r]$ are rV and $\binom{r}{2}V + r^2E$ respectively.

2. Any edge of G (along with its incident vertices) induces a K_{2r} in $G[r]$.

3. If $r = st$ then $G[r] = G[st] = G[s][t]$. Thus, for example, $K_1[st] = K_1[s][t] = K_s[t] = K_{st}$. In particular, for any complete graph K_s and any positive integer t, we have $K_s[t] = K_{st}$.

4. Independence is invariant under inflation. That is, if the independence number of G is α then the independence number of $G[r]$ is α as well.

5. If the clique number of G is ω, then the clique number of $G[r]$ is $r\omega$.

3 The Thickness of Cloned Graphs

In this section we will begin the study of cloned graphs. In the next section we will generalize to r-inflated graphs.

Example 1. Let P_3 be the path of length two with vertices labeled u, v, w in linear order. Denote the vertices of the clone of P_3 by $u_1, u_2, v_1, v_2, w_1, w_2$. Two plane drawings of the clone of P_3 are given in Figure 1.

The righthand drawing in Figure 1, a straightline drawing given by nesting non-convex K_4s, helps to illustrate the inductive argument that we will use next.

Proposition 1. If T is a tree, then its clone is planar.

Proof. If T is a single edge, then its clone is K_4, which is planar. Let T be a tree with at least two vertices and assume that all trees on fewer vertices than T are planar when cloned. Let w be a leaf of T with neighbor v. Let w_1, w_2 and v_1, v_2 be the vertices of $T[2]$ associated with vertices w and v. Prune w from T, yielding tree T'. Since T' has fewer vertices than T, its clone is planar by the inductive hypothesis. The distinction between $T'[2]$ and $T[2]$ is that $T'[2]$ is missing vertices w_1, w_2, the edge between them, and the edges between them and v_1, v_2. We wish to add these vertices and edges to $T'[2]$ without adding any crossings. Take a straightline plane drawing of $T'[2]$ and note that the (straightline) edge between v_1 and v_2 bounds two faces of T'_2. Add vertices w_1, w_2 to one of these faces so that they create a non-convex set with v_1, v_2. Now include the remaining edges among v_1, v_2, w_1, w_2, which may be accomplished without introducing crossings, thus providing a plane drawing of $T[2]$. □

The planarity of the clone of a tree is a useful tool for bounding the thickness of a cloned graph by using the arboricity of the original graph.

Proposition 2. If G has arboricity k, then $G[2]$ has thickness at most k.

Proof. Assume that G has arboricity k. Then every edge in G belongs to exactly one of k forests; denote these by $F_1, \ldots, F_k$. The forests provide a partition of the edges in G. For $i = 1, \ldots, k$ let H_i be the clone of the forest F_i. By (an obvious extension of) Proposition 1, each of the H_i is planar. Since every edge of G lives in precisely one forest, every edge of $G[2]$ connecting vertex-associated K_2s lives in precisely one of the forest clones. However, each edge of $G[2]$ that is a vertex-associated K_2 may exist in more than one H_i. Delete such extra edges from $H_2, \ldots, H_k$ yielding $H'_2, \ldots, H'_k$, all of which remain planar. Thus the graphs $H_1, H'_2, \ldots, H'_k$ provide a planar edge partition for $G[2]$ and therefore k bounds the thickness of $G[2]$. □

We compute the arboricity of the graphs in the upcoming examples and corollaries using the Nash-Williams formula [NW64], where the arboricity of a graph G is given by $\max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$. That is, the arboricity of G is obtained by finding the maximum arboricity of the over all subgraphs of G. For instance, any planar graph G has arboricity at most three since $\left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil \leq \left\lceil \frac{3|V(H)|-6}{|V(H)|-1} \right\rceil = \left\lceil 3 - \frac{3}{|V(H)|-1} \right\rceil \leq 3$ for all subgraphs $H \subset G$.

Example 2. The icosahedral graph I is planar and thus its arboricity is at most three. Using Nash-Williams, since $\lceil \frac{|E(I)|}{|V(I)|-1} \rceil = 3$ the arboricity is at least three. Hence I has arboricity exactly three. However, the clone of I has thickness two [GS09] and thus the bound in Proposition 2 is not always best possible.

Example 3. The Petersen graph P is non-planar, and hence its clone has thickness at least two. Further, since P has arboricity two and its clone is not planar, the clone of P has thickness two as well.

Example 4. The complete bipartite graphs $K_{3,3}, K_{3,4}, K_{3,5}$ and $K_{3,6}$ all have arboricity two. Since none of these graphs are planar, their clones are not planar and therefore their clones all have thickness precisely two.

Corollary 1. The clone of any outerplanar graph has thickness at most two. The clone of an outerplanar graph has thickness exactly two if and only if it is a not tree.

Proof. First we show that the arboricity of an outerplanar graph is at most two. Since $\frac{|E(G)|}{|V(G)|-1}$ is a ratio of edges to vertices, its maximum occurs when G is maximal outerplanar. Thus if G has n vertices we have $\left\lceil \frac{|E(G)|}{|V(G)|-1} \right\rceil \leq \left\lceil \frac{2n-3}{n-1} \right\rceil = \left\lceil 2 - \frac{1}{n-1} \right\rceil = 2$. Since any subgraph of an outerplanar graph is outerplanar, the bound on the ratio remains the same over all subgraphs H of G. Thus the arboricity of G is at most two and $G[2]$ has thickness at most two by Proposition 2.

Finally, it is easy to check that the clone of any cycle contains a K_5 subdivision and thus has thickness two. In that case, if an outerplanar graph contains a cycle, its clone must have thickness two. Moreover, since the clone of a tree is planar (Proposition 1), we have shown that the clone of an outerplanar graph has thickness two if and only if is not a tree. □

Corollary 2. The clone of a planar graph has thickness at most three.

Proof. Since planar graphs have arboricity at most three, applying Proposition 2 yields the result. □

Corollary 3. If a graph G has thickness t, then its clone has thickness at most $3t$.

Proof. Each planar layer has arboricity at most three, so if G has thickness t, it has arboricity at most $3t$. Then by Proposition 2 the thickness of $G[2]$ is also at most $3t$. □

4 The Thickness of r-inflated Graphs

We wish to study the thickness of r-inflated graphs. We start with the class of planar graphs.

Proposition 3. The r-inflation of an edge maximal planar graph on $n \geq 12$ vertices has thickness at least r. If the graph has more than 12 vertices, then the thickness is strictly greater than r.

Proof. Since G is edge maximal on n vertices, it has $3n-6$ edges. By Observation 1, $G[r]$ has rn vertices and $\binom{r}{2}n+r^2(3n-6)$ edges. Dividing the number of edges by the maximum number of edges in each planar layer yields a lower bound on the thickness of $G[r]$. This lower bound is:

$$\frac{7r^2n-12r^2-rn}{2(3rn-6)} = r + \frac{r^2n-12r^2-rn-12r}{6rn-12} = r + \frac{(n-12)(r^2-r)}{6rn-12}.$$

The second term is positive precisely when $n > 12$. Thus if $n > 12$ the thickness of $G[r]$ must be strictly greater than r. □

In Section 4.2 we will see that there is an edge maximal planar graph on 12 vertices (the icosahedral graph) whose r-inflation has thickness exactly r.

4.1 K_{2r} is the Disjoint Union of r Hamiltonian Paths

It is well known that for any integer $r \geq 1$, K_{2r+1} can be partitioned into r Hamiltonian cycles. See, for example, [Luc94, Als08]. In fact such a decomposition is a special case of the Oberwolfach Problem, which is described, for instance, in [HKR75]. A decomposition of K_{2r+1} into r Hamiltonian cycles leads easily to a decomposition of K_{2r} into r Hamiltonian paths. The latter is of use in decomposing certain classes of r-inflated graphs, which in turn gives a bound on their thickness. For this work, we require more than the existence of a Hamiltonian path decomposition for K_{2r}. We also need a certain measure of control over the endpoints of the Hamiltonian paths, as well as control over the pairs of vertices that are incident to the center-most edges of the paths. The purpose of the following lemma and corollary is to provide us with this control. The technique used in the proof has been used to generate a $2r \times 2r$ latin square, for example, in [Wil49], and is also similar to the methods used in [Til80] in a Hamiltonian path decomposition of a directed K_{2r}. For completeness, we give a detailed proof of the lemma.

Lemma 1. Let K_{2r} be a complete graph on $2r$ vertices with vertices labeled $v_1, \ldots, v_{2r}$. The edges of K_{2r} can be partitioned into r Hamiltonian paths, $P_1, P_2, \ldots, P_r$, each with one endpoint in $\{v_1, v_2, \ldots, v_r\}$ and the other in $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$.

Note that throughout the upcoming proof, and indeed this paper, we will use $\{1, 2, \ldots, 2r\}$ as a system of distinct representatives for arithmetic modulo $2r$.

Proof. Let P_1 be the Hamiltonian path of K_{2r} obtained by alternating vertices of the two sequences $v_1, v_2, \ldots, v_r$ and $v_{2r}, v_{2r-1}, \ldots, v_{r+1}$, beginning with vertex v_1. That is,

$$P_1 = v_1, v_{2r}, v_2, v_{2r-1}, \ldots, v_i, v_{2r-(i-1)}, v_{i+1}, \ldots, v_r, v_{r+1}.$$

Notice that the sequence of differences mod $2r$ between the indices of consecutive vertices from left to right in P_1 is given by $\{2i-1, 2r-2i : i = 1, \ldots, r\}$, which as a set is the same as $\{1, 2, \ldots, 2r-1, 2r\}$.

We generate a second path, P_2, by increasing the index of each vertex in P_1 by 1 (mod $2r$). That is,

$$P_2 = v_2, v_1, v_3, v_{2r}, \ldots, v_{i+1}, v_{2r-i+1}, v_{i+2}, \ldots, v_{r+1}, v_{r+2}.$$

And in general, path P_i is obtained by increasing the index of each vertex in P_1 by $i - 1$ (mod $2r$) for $i = 2, \ldots, r$.

We wish to show that these paths are edge disjoint. Note that P_i and P_j share an edge if and only if P_1 and P_{j-i+1} share an edge. By construction, the sequence of consecutive differences of vertex indices is invariant (mod $2r$). Thus the edges from vertex v_i to its righthand neighbors in the paths $P_1, \ldots, P_r$ are distinct. Similarly for its lefthand neighbors. The following will show that a righthand neighbor of v_i in P_1 cannot be a lefthand neighbor in some P_k.

There are two types of edges in P_1. One has the form $v_i v_{2r-i+1}$ and the other has the form $v_j v_{2r-j+2}$. We examine edges of the form $v_i v_{2r-i+1}$; a similar computation works for the other form. Suppose that v_{2r-i+1} is a righthand neighbor of v_i in P_1 and a lefthand neighbor of v_i in P_k. Since the edge $v_{2r-i+1}v_i$ in P_k is obtained by adding $k-1$ to the indices of an edge in P_1, it must have the form $v_{2r-j+k+1}v_{j+k-1}$ or $v_{j+k-1}v_{2r-j+k}$ in P_k. In the first case, this implies that $2r-i+1 = 2r-j+k+1$ and $i = j+k-1$. These combine to tell us that $2k = 1$, which is impossible since k is an integer. In the second case, we have $2r-i+1 = j+k-1$ and $i = 2r-j+k$, which combine to tell us that $k = 1$. Again, this is impossible because we are assuming $P_k \neq P_1$. Thus $v_{2r-i+1}v_i$ is not an edge of any P_k for $k \in \{2, \ldots, r\}$. A similar argument shows that $v_{2r-i+2}v_i$ is not an edge of any P_k, and thus paths P_i and P_j are edge-disjoint whenever $i \neq j$.

Finally, by design, the Hamiltonian paths $P_1, P_2, \ldots, P_r$, each have one endpoint in $\{v_1, v_2, \ldots, v_r\}$ and the other in $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$ and every vertex of the K_{2r} is an endpoint of a unique P_i. $\square$

Notice that in the proof of Lemma 1, if we were to replace a vertex v_i by its index i then such an enumeration of vertices would provide a graceful labeling of P_1 modulo $2r$. See [Wes01] for the definition of graceful labeling.

As mentioned in the introduction to this section, we want our Hamiltonian path decomposition of K_{2r} to fulfill specific requirements for the endpoints and for center-most edges of the paths. To clarify the nature of these requirements, we illustrate P_1, P_2, and P_3 for K_6 ($r = 3$).

$$P_1 = v_1\ v_6\ \underline{v_2\ v_5}\ v_3\ v_4$$

$$P_2 = v_2\ v_1\ \underline{v_3\ v_6}\ v_4\ v_5$$

$$P_3 = v_3\ v_2\ \underline{v_4\ v_1}\ v_5\ v_6$$

And for $r = 4$ we show the edge decomposition of K_8 into paths P_1, P_2, P_3, and P_4.

$$P_1 = v_1\ v_8\ v_2\ \underline{v_7\ v_3}\ v_6\ v_4\ v_5$$

$$P_2 = v_2\ v_1\ v_3\ \underline{v_8\ v_4}\ v_7\ v_5\ v_6$$

$$P_3 = v_3\ v_2\ v_4\ \underline{v_1\ v_5}\ v_8\ v_6\ v_7$$

$$P_4 = v_4\ v_3\ v_5\ \underline{v_2\ v_6}\ v_1\ v_7\ v_8$$

In Section 4.2 it will be important to know that the collection of edges at the center of each of the paths (endpoints shown underlined above) generated in Lemma 1 gives a perfect matching of K_{2r} for any $r \geq 2$.

Corollary 4. Let paths $P_1, P_2, \ldots, P_r$ be the path decomposition of K_{2r} as constructed in Lemma 1. A perfect matching for K_{2r} is given by the collection of center edges of the paths.

Proof. Since $2r$ is even, any path of length $2r-1$ has an edge at its center. In P_1, if r is odd, then the indices of the endpoints of the center edge are $\frac{r}{2}+\frac{1}{2}$ and $\frac{3r}{2}+\frac{1}{2}$. If r is even, then the indices of the endpoints of the center edge are $\frac{3r}{2}+1$ and $\frac{r}{2}+1$. Thus when r is odd the set of indices of the endpoints of the center edges is given by $\{\frac{r}{2}+\frac{1}{2}+j, \frac{3r}{2}+\frac{1}{2}+j : j=0,\ldots,r-1\}$ and when r is even, the set of indices of the endpoints of the center edges is given by $\{\frac{3r}{2}+1+j, \frac{r}{2}+1+j : j=0,\ldots,r-1\}$. It is straightforward to check that each set contains $2r$ distinct vertices. Thus the center edges of the P_is form a perfect matching for K_{2r}. □

4.2 The r-Inflation of the Icosahedral graph has thickness r

Recall from Proposition 3 that if G is any edge maximal planar graph on 13 or more vertices, the r-inflation of G has thickness at least $r+1$. Next we will show that the r-inflation of the icosahedral graph has thickness exactly r for any integer $r \geq 1$. This shows that the lower bound of r in Proposition 3 is best possible.

Theorem 5. The r-inflation of the icosahedral graph has thickness exactly r.

Proof. Let I be the icosahedral graph with vertex labeling and perfect matching as shown on the left in Figure 2.

Note that every edge in the perfect matching is the diagonal of a unique quadrilateral in I. Moreover, if we define a 5-wheel to be the join of a single vertex with a 5-cycle, then every vertex in I together with its five neighbors induces a 5-wheel. In fact, the subgraph induced by any vertex of I together with its five neighbors contains one matching edge on a spoke and one matching edge on its 5-cycle exactly as shown in Figure 2. This property is dependent on the particular perfect matching chosen. Thus for our edge decomposition that will lead to a thickness-r representation of $I[r]$, we describe a general construction for the r-inflation of a 5-wheel in which one edge of the perfect matching is on the 5-cycle and the other is one of the spokes. This construction will then be applied to the r-inflation of the entire icosahedral graph.

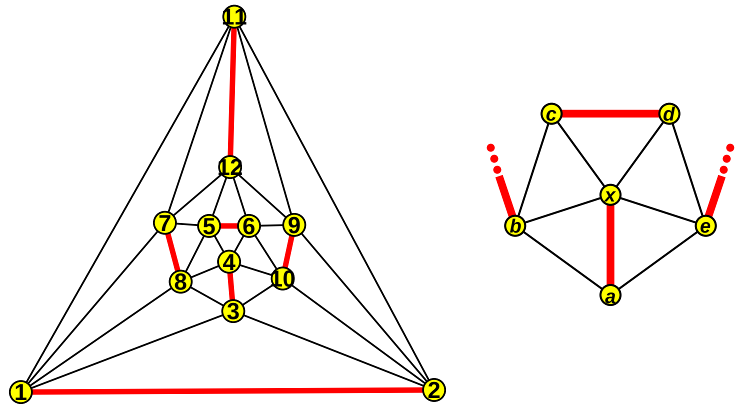

Figure 2: Chosen perfect matching for the icosahedral graph I (left), and subgraph in I induced by one vertex and its five neighbors (right)

Consider the two edges xa and cd of the given matching in Figure 2. By Observation 2, any edge in a graph G induces a K_{2r} in $G[r]$. In particular, each of xa and cd induce a K_{2r} in $I[r]$. We begin our construction by subdividing each of xa and cd with $2r-2$ vertices, which transforms each such edge into a path with $2r$ vertices. Let P_{xa} and P_{cd} be the paths resulting from the subdivisions of xa and cd respectively. In preparation for the use of Lemma 1, label the vertices of P_{xa} in linear order so that

$$P_{xa} = x_1, x_{2r}, x_2, x_{2r-1}, \ldots, x_i, x_{2r-(i-1)}, x_{i+1}, \ldots, x_r, x_{r+1},$$

where $x = x_1$ and $x_{r+1} = a$. Similarly, label the vertices in path P_{cd} in linear order so that

$$P_{cd} = c_1, c_{2r}, c_2, c_{2r-1}, \ldots, c_i, c_{2r-(i-1)}, c_{i+1}, \ldots, c_r, c_{r+1},$$

where $c = c_1$ and $c_{r+1} = d$.

Next add edges $\{x_i b, : i = 2, \ldots, r\}$, $\{x_i b, : i = r+2, \ldots, 2r\}$, $\{x_i e, : i = 2, \ldots, r\}$, $\{x_i e, : i = r+2, \ldots, 2r\}$, $\{x_1 c_i : i = 2, \ldots, r\}$, and $\{x_1 c_i : i = r+2, \ldots, 2r\}$. As previously mentioned, each edge of the given perfect matching uniquely determines a quadrilateral and therefore there is no ambiguity about which vertices of the 5-star are made adjacent to the vertices of the newly subdivided edges. Here xa determines $bxea$ and so b and e are made adjacent to all new vertices of P_{xa}. Further, the quadrilateral determined by cd includes vertex x, and so x is made adjacent to all new vertices of P_{cd}. By design, the transformed 5-wheel remains planar; see Figure 3. The construction, when applied to all of the edges of the given perfect matching of I, yields an edge maximal planar graph on $12r$ vertices, which we will denote by $\mathcal{L}_1$; see Figure 4. This "initial" planar graph together with the help of Lemma 1 will produce $r-1$ other graphs, each of which is isomorphic to $\mathcal{L}_1$ and whose union with $\mathcal{L}_1$ is contained in $I[r]$.

To this end, recall that in the proof of Lemma 1, a linearly ordered set of vertices $v_1, v_{2r}, \ldots, v_r, v_{r+1}$ represents a path of length $2r-1$ called P_1. We produced a set of $r-1$ more paths of length $2r-1$, called P_i, $i = 2, \ldots, r$, by effecting a permutation on the indices of the vertices in P_1. We showed that, by construction, the paths are pairwise edge disjoint. In particular, path P_i is obtained from P_1 by adding $i-1$ to the index of each vertex in P_1 and reducing the results modulo $2r$. For the problem at hand, we similarly obtain planar layer $\mathcal{L}_i$ from $\mathcal{L}_1$. That is, after each edge of the given matching of I has been subdivided, new edges added, and vertices suitably labeled to yield $\mathcal{L}_1$, we define $\mathcal{L}_i$ to be the graph obtained by adding $i-1 \pmod{2r}$ to the index of each vertex of $\mathcal{L}_1$. We do this for $i = 2, \ldots, r$. Since $\mathcal{L}_i$ is isomorphic to $\mathcal{L}_1$, each $\mathcal{L}_i$ is planar for $i = 1, \ldots, r$. Let $\mathcal{I} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_r$. It follows from Lemma 1 that $E(\mathcal{L}_i) \cap E(\mathcal{L}_j) = \varnothing$ whenever $i \neq j$.

For convenience, given any vertex x in I we let $x[r]$ denote the K_r in $I[r]$ associated with x. We will also need to keep track of which vertices of $\mathcal{L}_i$ are in the positions of the original vertices in I. Thus a vertex of $\mathcal{L}_i$ will be said to be in *position y* if it occupies the position y occupies in I. By Lemma 1, given a vertex x in I, for each vertex $v \in V(x[r])$, the label v will occupy position x in exactly one of the graphs $\mathcal{L}_i$ for $i = 1, \ldots, r$.

We claim that $\mathcal{I} = I[r]$. To prove the claim, we first take x to be an arbitrary vertex in I and consider the edges that must be incident with $x[r]$. See Figure 3 for a reminder of how x relates to its neighbors after subdividing edges of the given matching. Since the choice of x is arbitrary, it suffices to show that all of the edges in the join of $x[r]$ with $a[r]$, $b[r]$, $c[r]$, $d[r]$, and $e[r]$ are contained in $\mathcal{I}$.

The join of $x[r]$ with $a[r]$ induces a K_{2r} by definition. By our construction and Lemma 1, since xa is an edge of the chosen perfect matching, the join of $x[r]$ with $a[r]$ is contained in $\mathcal{I}$. Further, the join of $x[r]$ with $b[r]$ is contained in $\mathcal{I}$ because each vertex in $b[r]$ appears in position b in exactly one $\mathcal{L}_i$ by Lemma 1, and the vertex in position b is adjacent to all vertices in $x[r]$. The same argument applies to the join of $x[r]$ with $e[r]$. Finally, the

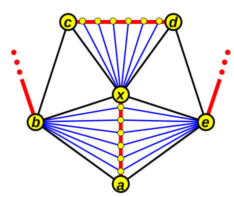

Figure 3: Subdivision of matching edges xa and cd by $2r-2$ vertices apiece, which induces two paths of length $2r-1$ in $I[r]$ with new edges added

same argument applies again to the position x: each vertex in $x[r]$ appears exactly once in position x, and a vertex in position x is adjacent to all elements of $c[r] \cup d[r]$. Thus far we have shown that $E(I[r]) \subset E(\mathcal{I})$. Since $\Theta(\mathcal{I}) \leq r$, we have $\Theta(I[r]) \leq r$.

Finally, since I is edge maximal on 12 vertices, Proposition 3 yields that $\Theta(I[r]) \geq r$. Altogether we have shown $\Theta(I[r]) = r$ and this completes the proof. $\square$

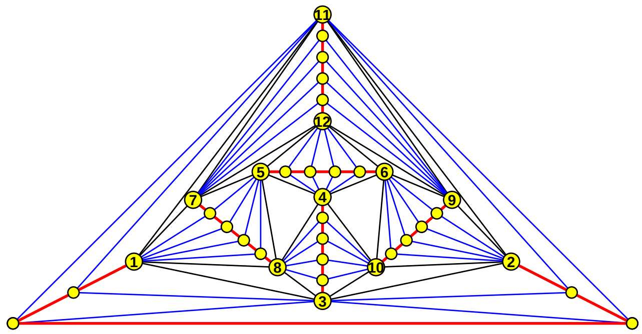

Figure 4: Subdivisions of matching edges and new edges carried through to all of the icosahedral graph I for $r = 3$.

In [BGS08] a *full permuted layer graph* is defined to be a thickness-two graph G for which there exist two isomorphic edge-disjoint planar subgraphs of G whose union is G. These particularly elegant graphs are built by using a base planar layer and then effecting a permutation of the vertices to achieve the second planar layer. The notion of full permuted layer graph adapts easily to the class of thickness-r graphs. Specifically, if H has thickness r and there exists a decomposition of H into r isomorphic planar graphs, then H is can be conceived of as a full permuted layer graph as well; these graphs are useful for constructing graphs of known thickness whose chromatic numbers can then be investigated. For example, by the proof of Theorem 5 we see that $I[r]$ is a full permuted layer graph. Moreover, In Section 5 we will show that the chromatic number of $I[r]$ can be precisely determined.

4.3 Example: A Thickness-4 Decomposition of $I[4]$

Using the vertex labels of the icosahedral graph I from Section 4.2, we inflate vertex i to a K_4 with vertices numbered by $\{i + 12j : j = 0, 1, 2, 3\}$ for $i = 1, \ldots, 12$. Under that vertex labeling scheme and with the techniques given in Theorem 5 we effect a thickness-4 decomposition of $I[4]$, which is shown in Figure 5.

Many of the ideas that were used in showing that the r-inflated icosahedral graph has thickness exactly r can be used to show that the thickness of the r-inflation of any tree is at most $\lceil \frac{r}{2} \rceil$. This will be accomplished in the next subsection.

4.4 The r-Inflation of a Tree has Thickness at Most $\lceil \frac{r}{2} \rceil$

Theorem 6. If T is a tree, then the thickness of $T[r]$ is at most $\lceil \frac{r}{2} \rceil$. Equivalently, both $T[2r]$ and $T[2r-1]$ have thickness at most r. Furthermore, for each fixed value of r, for all but finitely many trees, $T[2r]$ has thickness exactly r.

Proof. For the first claim, since $T[2r-1]$ is a subgraph of $T[2r]$, it suffices to show that $\Theta(T[2r]) \le r$. The proof is by construction; that is for any tree T we produce an edge

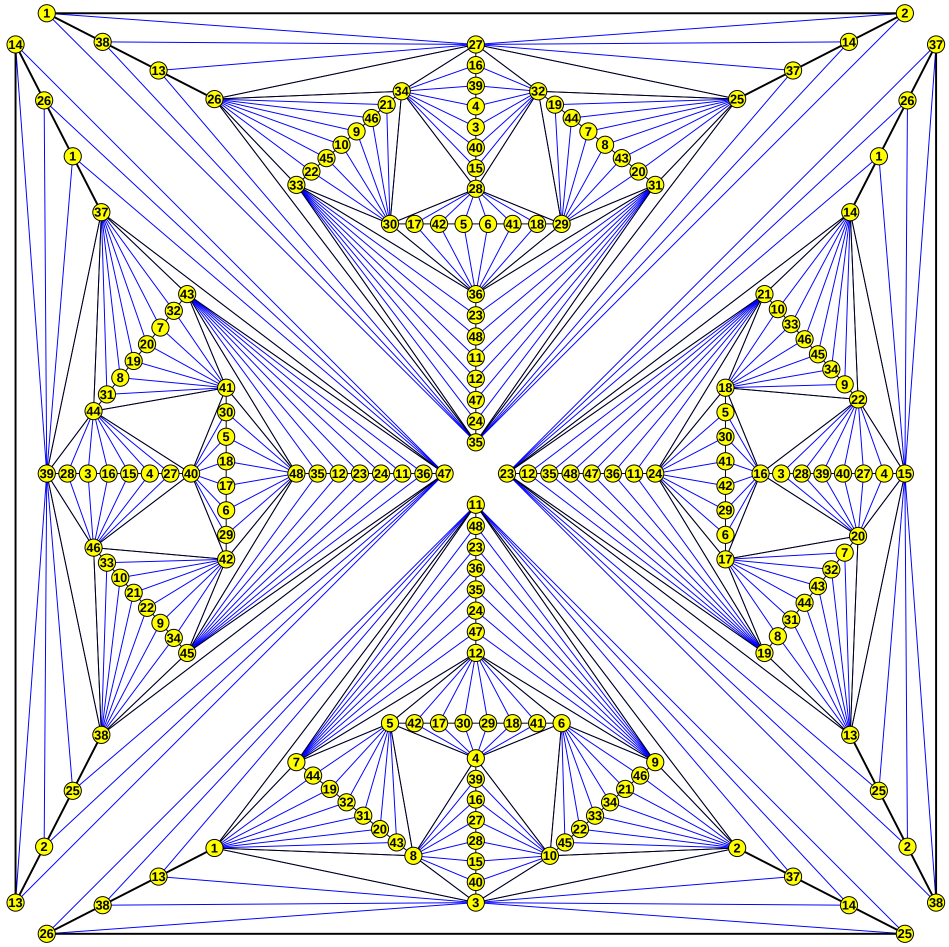

Figure 5: The four planar layers of a thickness-4 decomposition of $I[4]$.

partition of $T[2r]$ that induces r planar graphs. The construction is accomplished first on an n-star, $K_{1,n}$. From this, one can easily piece together the r planar layers of an appropriate set of stars to build the r-inflation of any tree.

First we recall that $K_{1,n}[2][r] = K_{1,n}[2r]$ by Observation 3. We begin the construction with the clone of $K_{1,n}$, which, by Proposition 1, is planar. Since each vertex of $K_{1,n}$ is transformed into an edge (a K_2), the set of 2-inflated vertices is a perfect matching in $K_{1,n}[2]$. The cardinality of the perfect matching is $n+1$, where one edge is associated with the vertex of degree n in $K_{1,n}$ and the other n edges are associated with the leaves. Denote the edges of the matching that are associated with the leaves by $\ell_1, \ell_2, \ldots, \ell_n$, and denote the edge of the matching associated with the vertex of degree n by *CenterEdge*. Figure 6 (b) shows a straightline embedding of $K_{1,n}[2]$ for $n = 4$. This method works for any $n \geq 1$.

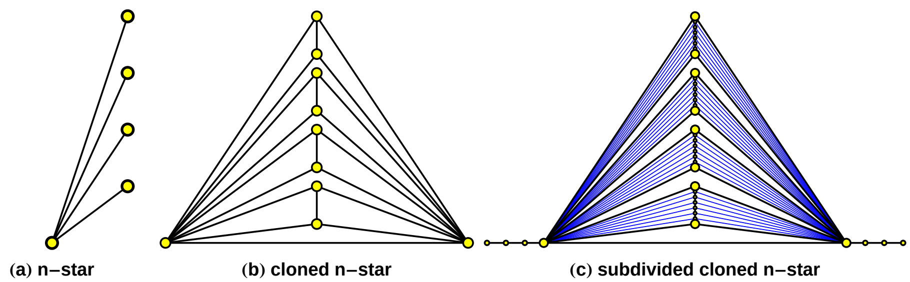

Figure 6: Evolution of the $2r$-inflation of a star

The vertex of degree n is an endpoint of every edge of $K_{1,n}$. Thus *CenterEdge* is a common edge in the K_4s resulting from inflating the edges of $K_{1,n}$.

The next step is similar to the construction given in the proof of Theorem 5. In particular, we will construct $K_{1,n}[2r]$ from $K_{1,n}[2]$ by subdividing the edges of the given perfect matching of $K_{1,n}[2]$ with $2r-2$ vertices each and then adding some new edges, thereby producing an initial planar layer. A suitable set of $r-1$ permutations of the index labels of the subdivision vertices will yield $r-1$ more planar layers of an edge decomposition for $K_{1,n}[2r]$.

Consider any edge ℓ_i of the given matching of $K_{1,n}[2]$. Subdivide ℓ_i with $2r-2$ vertices, which together with the endpoints of ℓ_i, induces a path of length $2r-1$. In preparation for the use Lemma 1 we label these $2r$ vertices by

$$v_1, v_{2r}, v_2, v_{2r-1}, \ldots, v_i, v_{2r-(i-1)}, v_{i+1}, \ldots, v_r, v_{r+1}.$$

Similarly, take the edge *CenterEdge* and subdivide it with $2r-2$ vertices. Label the vertices of the resulting path of length $2r-1$ in linear order by

$$c_1, c_{2r}, c_2, c_{2r-1}, \ldots, c_i, c_{2r-(i-1)}, c_{i+1}, \ldots, c_r, c_{r+1}.$$

Suppose the endpoints of the center-most edge of subdivided *CenterEdge* are x and y. (The vertices x and y are of the form c_i and c_j, the indices are known, differ depending on whether r is odd or even, and are unimportant for now.) Transform $K_{1,n}[2]$ further by adding edges $\{xv_j : j = 2, \ldots, r\} \cup \{xv_j : j = r+1, \ldots, 2r\} \cup \{yv_j : j = 2, \ldots, r\} \cup \{yv_j : j = r+1, \ldots, 2r\}$. The resulting graph, denoted by $\mathcal{T}_1$, is planar; see Figure 6 (c).

To achieve $r-1$ additional planar layers, we permute the vertex labels of $\mathcal{T}_1$. Specifically, for $i = 2, \ldots, r$, define $\mathcal{T}_i$ to be the graph obtained by adding $i-1 \pmod{2r}$ to the index of each vertex in $\mathcal{T}_1$. Observe that by design $\mathcal{T}_i$ is isomorphic to $\mathcal{T}_1$ for $i = 1, \ldots, r$, which means that each $\mathcal{T}_i$ is planar.

Penultimately, to see why $\Theta(K_{1,n}[2r]) \leq r$, it suffices to show that $K_{1,n}[2r] \subset \bigcup_{i=1}^{r} \mathcal{T}_i$. By Lemma 1 we know that the construction accounts for all the edges of the K_{2r} induced by each leaf vertex in $K_{1,n}$. Moreover, by Corollary 4 the endpoints of the center-most edge of the subdivision of *CenterEdge* together with all of the corresponding endpoints in the vertex-permuted graphs $\mathcal{T}_i$ for $i = 1, \ldots, r-1$ induce a perfect matching of the K_{2r} induced by the vertex of degree n in $K_{1,n}$. Hence, every vertex in this K_{2r} is adjacent to all of the inflations of leaf vertices in $K_{1,n}[2r]$. Thus $K_{1,n}[2r] \subset \bigcup_{i=1}^{r} \mathcal{T}_i$, which is what we needed to show.

Finally, since any tree T is an edge disjoint union of stars, we may construct an edge decomposition of $T[2r]$ into r planar graphs as follows. Let $Star_1 \cup \cdots \cup Star_q$ be a star decomposition of T. Note that since T is a tree, two distinct stars $Star_i$ and $Star_j$ intersect in at most one vertex. Moreover, this decomposition can be arranged so that if $Star_i$ and $Star_j$ do share a vertex ℓ, then ℓ is the center vertex of $Star_i$ and a leaf vertex of $Star_j$. Suppose that $Star_j[2r]$ has been decomposed into r planar graphs as described in the first part of this proof. In $Star_j[2r]$, leaf vertex ℓ is transformed into a path, say ℓ', with $2r$ vertices. Let x and y be the endpoints of the centermost edge of ℓ'. See Figure 7. Consider the triangular region contained in the 3-cycle induced by x, y, and one endpoint z of *CenterEdge* in $Star_j[2r]$. One planar layer of the construction of $Star_i[2r]$ can then take place in this triangular region (shaded in Figure 7) using vertices x and y (but not z). The remaining $r-1$ layers of $Star_j[2r]$ are attained by the standard set of permutations of the vertex labels of the first planar layer. Continue this process until all of $T[2r]$ has been constructed. Hence, $\Theta(T[2r]) \leq r$.

Finally, let $r \in \mathbb{Z}^+$ be fixed and suppose $|V(T)| = n$ (and hence $|E(T)| = n-1$). By Observation 1, $T[2r]$ has $2rn$ vertices and $n\frac{2r(2r-1)}{2} + 4r^2(n-1) = 6r^2 - 4r^2 - rn$ edges. A planar graph on $2rn$ vertices has at most $6rn - 6$ edges in which case the thickness of $T[2r]$ is at least $\left\lceil \frac{6r^2-4r^2-rn}{6rn-6} \right\rceil$, which tends to $\left\lceil r - \frac{1}{6} \right\rceil = r$ as n tends to ∞. Thus for each fixed value of r, we have $\Theta(T[2r]) \geq r$ for all but finitely many trees. This completes the proof. $\square$

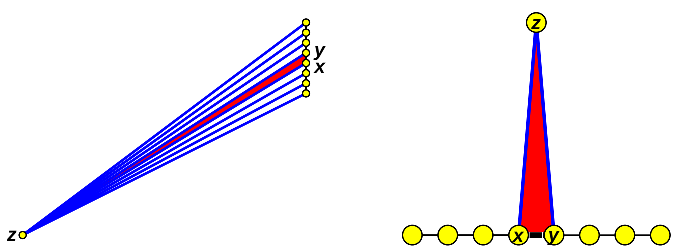

Figure 7: One planar layer of a new $2r$-inflated star can be constructed entirely inside the shaded region including edge xy.

Proposition 2 in Section 3 gives a bound on the thickness of a cloned graph $G[2]$ by way of the arboricity of G. We close this section by stating an analogous proposition that bounds the thickness of any r-inflated graph, the proof of which is nearly identical to that of the proof of Proposition 2.

Proposition 4. If G has arboricity k, then $G[r]$ has thickness at most $k\left\lceil \frac{r}{2} \right\rceil$.

We end this section with a natural generalization of Corollary 3.

Proposition 5. If G has thickness t, then $G[2r]$ has thickness at most $3tr$.

Proof. The edges of any graph with thickness t can be partitioned into t planar graphs, each of which has arboricity at most three. The $2r$-inflation of each of the (at most) three forests has thickness at most r (Corollary 3), from which the result follows. $\square$

Let us do a brief analysis on the bound given by Proposition 5. Recall that if $n \neq 9, 10$ the thickness of K_n is exactly $\lfloor \frac{n+7}{6} \rfloor$ [AG76]. Let $G = K_n$ and for the sake of asymptotics suppose $n \geq 12$ and $6|n$. By Observation 3, $G[r] = K_{nr}$. Then Proposition 5 bounds the thickness of K_{nr} by $3(\frac{n}{6}+1)\frac{r}{2} = \frac{rn}{4} + \frac{3r}{2}$. On the other hand, K_{nr} has thickness exactly $\frac{rn}{6}+1$. Since $\lim_{r\to\infty} \frac{rn}{4} + \frac{3r}{2} - (\frac{rn}{6}+1) = \infty$, the bound in Proposition 5 is not especially good.

5 Chromatic Numbers of r-inflated Graphs

It is straightforward to give an upper bound for the chromatic number of $G[r]$ by way of the chromatic number of G. This result is given in the next proposition. Further, we will see that the upper bound can be achieved for every $r \in \mathbb{Z}^+$ by the graph $I[r]$.

Proposition 6. If $\chi(G) = k$, then $\chi(G[r]) \leq rk$.

Proof. Suppose that $\chi(G) = k$. Consider the set of rk colors $\{1_1, \ldots, 1_r, 2_1, \ldots, 2_r, \ldots, k_1, \ldots, k_r\}$. If vertex v in G is colored s in a proper k-coloring of G then color the associated vertices of $G[r]$ with $s_1, \ldots, s_r$. There are two types of adjacent vertices in $G[r]$, those of the form v_i, v_j where both are associated with the vertex v in G but $i \neq j$, and those of the form v_i, u_j where the associated vertices v and u are adjacent in G. If the color of v in G is s then colors of v_i and v_j in $G[r]$ are s_i and s_j. Since $i \neq j$, these are different colors. If u and v are adjacent in G and are colored s and t, then $s \neq t$ which implies that $s_i \neq t_j$. Thus we have described a proper rk-coloring of $G[r]$. We conclude that if $\chi(G) = k$, then $\chi(G[r]) \leq rk$. $\square$

Example 5. The icosahedral graph I is 4-chromatic and has independence number three. By Observation 4, $I[r]$ also has independence number three and thus its independence ratio is $\frac{3}{12r} = \frac{1}{4r}$. Since the chromatic number is at least the inverse of independence ratio, the chromatic number of the r-inflation is at least $4r$. By Proposition 6 it is also at most $4r$. Thus $\chi(I[r]) = 4r$. Recall that by Theorem 5 we also know that the thickness of $G[r]$

is precisely r. Thus $\{I[r]\}_{r=1}^{\infty}$ provides a rare substantive family of graphs for which we can determine the precise thickness and the precise chromatic number. (See [BGS08] for another such example.) Even more, the example shows that the largest possible chromatic number for an r-inflated planar graph is achieved for every positive integer r.

Proposition 7. If the clique number $\omega(G)$ equals the chromatic number $\chi(G)$, then $\chi(G[r]) = r\chi(G)$.

Proof. Suppose that $\chi(G) = \omega(G)$. By Observation 5, $\omega(G[r]) = r\omega(G)$. But $\omega(G[r]) \leq \chi(G[r])$, so $r\omega(G) = \omega(G[r]) \leq \chi(G[r])$. By our assumption we can replace $\omega(G)$ with $\chi(G)$ so $r\chi(G) \leq \chi(G[r])$. However, by Proposition 6 we know that $\chi(G[r]) \leq r\chi(G)$. Thus we have equality. $\square$

The following general result of Gao and Zhu [GZ96] allows us to easily compute the chromatic number of the inflation of an odd cycle (see also [Cat79]).

Theorem 7. [GZ96] If $\chi(H) = r$, then $\chi(C_{2k+1}[H]) = 2r + \lceil \frac{r}{k} \rceil$.

Thus $\chi(C_{2k+1}[r]) = 2r + \lceil \frac{r}{k} \rceil$. In particular the clone of any odd cycle of length greater than three has chromatic number precisely five, while the 3-inflation of any odd cycle greater than five has chromatic number precisely seven. In [BGS08] we also saw that $C_{2k+1}[3]$ has thickness two for all $k \geq 2$.

In this article we have delved into the thickness of r-inflated graphs and have given a first look at the chromatic number of r-inflated graphs, as well. Further results on both the thickness and chromatic number of cloned and r-inflated graphs will follow in a sequel.

6 Open Problems

The results in this paper suggest the following open problems.

1. Given a planar graph G, we have seen that the thickness of $G[2]$ is one of 1, 2, or 3. Further, $\Theta(G[2]) = 1$ if and only if G is a tree (because the clone of a cycle contains a subdivision of K_5). Characterize the planar graphs G for which $\Theta(G[2]) = 2$. For which $\Theta(G[2]) = 3$.

2. A natural generalization of r-inflation would be to allow the flexibility to replace a vertex with a complete graph of any size, whereupon the independence number is still preserved. Suppose G is a planar graph on n vertices labeled $1, \ldots, n$ in some order; then let $G[s_1, s_2, \ldots, s_n]$ denote the graph G in which vertex i has been inflated to a K_{s_i} (with edges achieved by taking the join of neighboring complete graphs, as usual). If there is at least one pair (i, j) for which $s_i \neq s_j$, can the best upper bound for the chromatic number be achieved as it was for $G[r]$? What can be said more generally about the thickness and chromatic number of $G[s_1, s_2, \ldots, s_n]$?

3. What is the thickness of $C_{2k+1}[r]$ for any $k \geq 4$ and any $r \geq 4$? Note that $K_9 = C_3[3]$ has thickness three [Tut63] and is an exceptional case in the formula for the thickness of the complete graph [AG76].

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