1 Introduction and Results

Consider a collection of random variables $\{Y_{i,j,k} : (i,j,k) \in (\mathbb{Z}^+)^2 \times \mathbb{N},\ i \neq j\}$ on an underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that $\sup_{i,j,k,\omega} |Y_{i,j,k}(\omega)| \leq C < \infty$, $EY_{i,j,k} = 0$, and $Y_{i,j,0} = 0$ for all $(i,j) \in (\mathbb{Z}^+)^2,\ i \neq j$. For all $(\bar{i}, \bar{j}, \bar{k}),\ (\underline{i}, \underline{j}, \underline{k}) \in \{(i,j,k) \in (\mathbb{Z}^+)^3,\ i \neq j,\ k \geq 1\}$, assume that

$$Cov(Y_{\bar{i},\bar{j},\bar{k}}, Y_{\underline{i},\underline{j},\underline{k}}) = \begin{cases} \sigma^2 & : \ \bar{i} = \underline{i},\ \bar{j} = \underline{j},\ \bar{k} = \underline{k} \\ \gamma & : \ \bar{i} = \underline{i},\ \bar{j} = \underline{j},\ \bar{k} \neq \underline{k} \\ \rho & : \ \bar{i} = \underline{i},\ \bar{j} \neq \underline{j}, \\ \tau & : \ \bar{i} \neq \underline{i},\ \bar{j} = \underline{j} \\ \nu & : \ \bar{i} = \underline{j},\ \bar{j} \neq \underline{i} \\ \eta & : \ \bar{j} = \underline{i},\ \bar{i} \neq \underline{j}, \\ \xi_1 & : \ \bar{i} = \underline{j},\ \bar{j} = \underline{i},\ \bar{k} = \underline{k} \\ \xi_2 & : \ \bar{j} = \underline{i},\ \bar{i} = \underline{j},\ \bar{k} \neq \underline{k}, \end{cases}$$

else $Y_{\bar{i},\bar{j},\bar{k}}, Y_{\underline{i},\underline{j},\underline{k}}$ are independent. Note that $\nu = \eta$, and, for example, $E(Y_{1,2,\bar{k}}, Y_{1,3,\underline{k}}) = \rho$.

Let $\{M_i : i \in \mathbb{Z}^+\}$ either be a collection of i.i.d. $\mathbb{N}$-valued random variables or an ergodic collection of random variables on $(\Omega, \mathcal{F}, \mathcal{P}\}$ with $0 < E(M_1) = \alpha < \infty$, $E((M_1)^2) = \beta < \infty$, $E((M_1)^j) < \infty$, $1 \leq j \leq 4$. For $i,\ j \in \mathbb{Z}^+$, define $M_{i,j} = M_i M_j$. Assume that $\{M_i : i \in \mathbb{Z}^+\}$ is independent of $\{Y_{i,j,k} : (i,j,k) \in (\mathbb{Z}^+)^2 \times \mathbb{N},\ i \neq j\}$. Take note that $M_{i,j}$ may be equal to zero. With this in mind, we will now make a convention regarding notation for sums and products.

Convention: Let $a < b,\ a,\ b \in \mathbb{Z}$, and $\{o_i\}_{i \in \mathbb{Z}}$ be an collection of real numbers. Then,

$$\sum_{i=b}^{a} o_i = 0,\ \prod_{i=b}^{a} o_i = 1.$$

We are interested in proving a central limit theorem for $\{Y_{i,j,k} : (i,j,k) \in (\mathbb{Z}^+)^2 \times \mathbb{N},\ i \neq j,\ 0 \leq k \leq M_{i,j}\}$. To be explicit, in this case we prove the following result.

Theorem 1.1

$$\frac{1}{n^{3/2}} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{k=1}^{M_{i,j}} Y_{i,j,k} \xrightarrow{d} N(0, \alpha^2 \beta (\rho + \tau + \nu + \eta)).$$

Note that the normalizing constants $\{n^{3/2}\}_{n \geq 1}$ are different than what is to be expected from a standard central limit theorem, which in this case would be $\{n\}_{n \geq 1}$. We now prove a lemma to help us estimate the parameters of interest.

Let $g_0, g_1 : \mathbb{N}^2 \to \mathbb{N}$ such that for $(x,y) \in \mathbb{N}^2$, $g_0(x,y) = x$, $g_1(x,y) = y$. Let $h_0, h_1 : \mathbb{N}^2 \to \mathbb{N}^2$ such that for $(x,y) \in \mathbb{N}^2$, $h_0(x,y) = (x,y)$, $h_1(x,y) = (y,x)$. Also, let $T = \{t_1 = \rho, t_2 = \tau, t_3 = \nu, t_4 = \eta\}$, and

$$\begin{aligned}\langle a(t_1), b(t_1)\rangle &= \langle 0,0\rangle, \\ \langle a(t_2), b(t_2)\rangle &= \langle 1,1\rangle, \\ \langle a(t_3), b(t_3)\rangle &= \langle 0,1\rangle, \\ \langle a(t_4), b(t_4)\rangle &= \langle 1,0\rangle \quad .\end{aligned}$$

It now may be observed that for $1 \le l \le 4$, $i,j,k,s_1,s_2 \in \mathbb{Z}^+$

$$E(Y_{i,j,s_1} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2}) \quad = \quad t_l \quad .$$

With this in mind, for $1 \le l \le 4$, define

$$\hat{V}_{l,n} \quad = \quad \frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \sum_{s_1=1}^{M_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{M_{h_{b(t_l)}(g_{a(t_l)}(i,j),k)}} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2} \quad .$$

For example, we have that

$$\hat{V}_{1,n} \quad = \quad \frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \sum_{s_1=1}^{M_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{M_{i,k}} Y_{i,k,s_2} \quad .$$

We are now in position to state our second theorem.

Theorem 1.2

$$\frac{1}{\sqrt{\sum_{i=1}^{4} \hat{V}_{i,n}}} \frac{1}{n^{3/2}} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \sum_{k=1}^{M_{i,j}} Y_{i,j,k} \xrightarrow{d} N(0,1).$$

2 Proof of Theorem 1.1

Let $t \in \mathbb{R}$. We will show that

$$E \exp\{it \frac{1}{n^{3/2}} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \sum_{k=1}^{M_{i,j}} Y_{i,j,k}\} \to e^{-\alpha^2 \beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.$$

We break the proof of this up into steps defined by propositions some of whose proofs, due to length, are found in Section 4.

2.1 Step 1:

Let

$$\Omega_1 = \{\omega \in \Omega : \lim_n \frac{1}{n}\sum_{i=1}^{n} M_i(\omega) = \alpha,\ \lim_n \frac{1}{n}\sum_{i=1}^{n}(M_i(\omega))^2 = \beta,$$
$$\lim_n \frac{1}{n}\sum_{i=1}^{n}(M_i(\omega))^j = E((M_1)^j) < \infty,\ 1 \le j \le 4\}.$$

Proposition 2.1 *Let $t \in \mathbb{R}$. If*

$$E\exp\{t\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \to e^{\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}$$

for all $\{m_i\}_{i=1}^{\infty} \in \{M_i(\Omega_1)\}_{i=1}^{\infty}$, then

$$E\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}} Y_{i,j,k}\} \to e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.$$

Proof. See Section 4. **Q.E.D.**

Assumption: Henceforth, it will be assumed that $\{m_i\}_{i=1}^{n} \in \{M_i(\Omega_1)\}_{i=1}^{n}$. We also assume from here forth that n is so large that $m_i,\ m_j > 0$ for some $1 \le i \neq j \le n$, which can be done, as $\alpha > 0$, so the distribution of the M_i cannot be degenerate at zero.

Proposition 2.2

$$\begin{aligned} E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+\frac{1}{n^{3/2}}tY_{i,j,k})\} &\le E\exp\{t\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \\ &\le (1+o_n(1))E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+\frac{1}{n^{3/2}}tY_{i,j,k})\}. \end{aligned}$$

Proof. See Section 4. **Q.E.D.**

Remark 2.1 We now show that

$$E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+\frac{1}{n^{3/2}}tY_{i,j,k})\} \to e^{\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.$$

By Proposition 2.2, this will suffice to prove Theorem 1.1.

2.2 Step 2:

We first establish some notation and results from graph theory.

2.2.1 Graph Theory Excursion I

Let $n \in \mathbb{Z}^+$. Define

$$\begin{aligned}
\mathbb{Z}^3_{e,+} &= \{\langle i,j,k\rangle \in (\mathbb{Z}^+)^3 : i \neq j,\ 1 \leq m_{i,j},\ 1 \leq k \leq m_{i,j}\},\\
\mathbb{Z}^3_{1,+} &= \{\langle i,j,k\rangle \in \mathbb{Z}^3_{e,+} :\ k = 1\},\\
[n]^3_e &= \{\langle i,j,k\rangle \in \mathbb{Z}^3_{e,+} :\ i,\ j \leq n\},\\
[n]^3_1 &= \{\langle i,j,k\rangle \in [n]^3_e :\ k = 1\}.
\end{aligned}$$

If $G \subset \mathbb{Z}^3_{e,+}$, we refer to G as a graph, and any $(i,j,k) \in \mathbb{Z}^3_{e,+}$ as a point or individual. For $n \in \mathbb{Z}^+$, if $G \subset [n]^3_e$, we say G lives on $[n]^3_e$, or is on $[n]^3_e$. If G is such that $|G| = \theta \in \mathbb{Z}^3_{e,+}$, we will call G a θ-graph.

The empty set, the graph that contains no points, will be called the empty graph, and will be denoted as $\emptyset$. We call G a hyper-row, and in particular for $i' \in \mathbb{Z}^+$, the i'th hyper-row iff

$$G = \{(i,j,k) \in \mathbb{Z}^3_{e,+} : i = i'\}.$$

Similarly, call G a hyper-column, and in particular for $j' \in \mathbb{Z}^+$, the jth hyper-column iff

$$G = \{(i,j,k) \in \mathbb{Z}^3_{e,+} : j = j'\}.$$

If G_1 is a hyper-row, and G_2 is a hyper-column, we call the graph H a cylinder iff $H \subset G_1 \cap G_2$. Therefore, a non-empty cylinder $\mathbf{x}$ takes the form $\mathbf{x} = \{(i,j,k) \in \mathbb{Z}^3_{e,+} : i = i',\ j = j',\ k \in G\}$ for some $(i',j',1) \in Z^3_{1,+}$, $G \subset \{1,\ldots,m_{i',j'}\}$. A graph $G \in \mathbb{Z}^3_{e,+}$ that is a cylinder will be called cylindrical, and a graph $G \in \mathbb{Z}^3_{e,+}$ that is not a cylinder will be called non-cylindrical.

We call a graph $G \in \mathbb{Z}^3_{e,+}$ balanced if for each point $(i,j,k) \in G$, we have $m_{j,i} > 0$, and $(j,i,l) \in G$ for some $1 \leq l \leq m_{j,i}$. We call a graph $G \in \mathbb{Z}^3_{e,+}$ unbalanced if for some point $(i,j,k) \in G$, we have $m_{j,i} = 0$, or $(j,i,l) \notin G$ for all $1 \leq l \leq m_{j,i}$.

For a graph $G \in \mathbb{Z}^3_{e,+}$, we define the projection of G as $Proj(G)$ where

$$Proj(G) = \{(i,j,1) \in \mathbb{Z}^3_{1,+} :\ (i,j,k) \in G \text{ for some } 1 \leq k \leq m_{i,j}\}$$

In addition, for any $G \in \mathbb{Z}^3_{1,+}$, $H \in \mathbb{Z}^3_{e,+}$, we define the inverse projection of G onto H as $Proj^{-1}_G(H)$ where

$$Proj^{-1}_G(H) = \{(i,j,k) \in H :\ (i,j,1) \in G\}.$$

If $A,\ B$ are finite subsets of $\mathbb{Z}^+$, then we call $A \times B \times \mathbb{Z}^+ \cap \mathbb{Z}^3_{e,+}$ an array, and to be specific, an $(|A| \times |B|)$-array. We call an $(|A| \times |B|)$-array complete or a complete $(|A| \times |B|)$-array iff $m_{i,j} > 0$ for $i \in A,\ j \in B,\ i \neq j$. Note that an array is the intersection of a union of hyper-rows and a union of hyper-columns.

If $\kappa \in \mathbb{Z}^+$, σ a permutation of $\{1, \ldots, \kappa\}$, and $G_1, \ldots, G_\kappa$ are κ distinct graphs, then we will let $\langle G_{\sigma(1)}, \ldots, G_{\sigma(\kappa)} \rangle$ be the ordered collection of $G_1, \ldots, G_\kappa$ corresponding to σ, and $\{G_{\sigma(1)}, \ldots, G_{\sigma(\kappa)}\}$ the unordered collection of $G_1, \ldots, G_\kappa$. Thus, for distinct $\sigma_1,\ \sigma_2$ permutations of $\{1, \ldots, \kappa\}$, we have that

$$\begin{aligned} \langle G_{\sigma_1(1)}, \ldots, G_{\sigma_1(\kappa)} \rangle &\neq \langle G_{\sigma_2(1)}, \ldots, G_{\sigma_2(\kappa)} \rangle, \\ \{G_{\sigma_1(1)}, \ldots, G_{\sigma_1(\kappa)}\} &= \{G_{\sigma_2(1)}, \ldots, G_{\sigma_2(\kappa)}\}. \end{aligned}$$

For any $1 \leq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} m_{i,j} = |[n]^3_e|$, we denote $\mathbf{A}^n_\kappa$ as the set of all κ-graphs that live on $[n]^3_e$, that is

$$\mathbf{A}^n_\kappa = \Big\{ G \subset [n]^3_e : |G| = \kappa \Big\}.$$

2.2.2 Step 2 Proposition

Proposition 2.3

$$E\{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \prod_{k=1}^{m_{i,j}} (1 + \frac{1}{n^{3/2}} t Y_{i,j,k})\} = 1 + \sum_{\kappa=1}^{|[n]^3_e|} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in \mathbf{A}^n_\kappa} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}}.$$

Proof. This is obvious. **Q.E.D.**

2.3 Step 3:

We continue our excursion into graph theory.

2.3.1 Graph Theory Excursion II

For $(q, r, s), (u, v, w) \in [n]^3_e$, we write $(q, r, s) \leftrightarrow (u, v, w)$ iff $i = j$ for some $i \in \{q, r\}$ and some $j \in \{u, v\}$, and say that (q, r, s) and (u, v, w) touch. If (q, r, s) and (u, v, w) do not touch, we write $(q, r, s) \not\leftrightarrow (u, v, w)$. From the dependence structure of our array of random variables, this means that $(q, r, s) \not\leftrightarrow (u, v, w)$ iff $Y_{(q,r,s)}$ and $Y_{(u,v,w)}$ are independent. We will say that two distinct graphs G_1 and G_2 touch if there exists an $x \in G_1$ and a $y \in G_2$ such that $x \leftrightarrow y$. We make the convention that the empty graph touches all graphs. Finally, for any finite graph $G \in [n]^3_e$, we will define

$$V_t(G) = \{(i, j, k) \in [n]^3_e : \ (i, j, k) \leftrightarrow (q, r, s) \text{ for some } (q, r, s) \in G\}.$$

If $1 \le l \le |[n]_e^3|$, and $G_1, \ldots, G_l \subset [n]_e^3$, we call $G_1, \ldots, G_l$ *mutually separated* iff there does not exist distinct $i, j \in \{1, \ldots, l\}$ such that G_i and G_j touch. Thus, if two graphs are not mutually separated, they touch. Note that two graphs that are mutually separated are mutually disjoint; however, the converse is not true. Also, note that again by the dependence structure of our array of random variables, if $G_1, \ldots, G_l$ are mutually separated, then $\{Y_{(i,j,k)}\}_{(i,j,k)\in G_1}, \ldots, \{Y_{(i,j,k)}\}_{(i,j,k)\in G_l}$ are mutually independent collections of random variables, although the marginal random variables of any one collection need not be independent.

If $x,\ y \in G$, we say that x and y communicate in G, written $x \overset{G}{\Leftrightarrow} y$, iff x and y touch or there is a sequence of distinct points $z_1, \ldots, z_l \in G$, $1 \le l \le |G| - 2$, such that

$$x \leftrightarrow z_1 \cdots \leftrightarrow z_l \leftrightarrow y.$$

For a graph G on the $[n]_e^3$, we will call G a connected graph iff all points of G communicate in G. Thus, a connected graph cannot be partitioned into a collection of mutually separated subgraphs. More to the point, if G is a connected graph, $\{Y_{i,j,k}\}_{(i,j,k)\in G}$ cannot be partitioned into any mutually independent sub-collections of random variables.

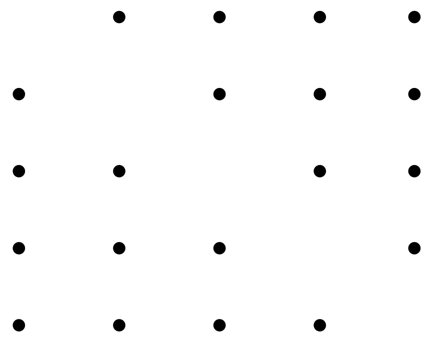
For $\theta \in \mathbb{Z}^+$, we say that G is a θ-connected graph iff G is connected and $|G| = \theta$. Note that if G_1 and G_2 are two connected graphs that touch, then $G_1 \cup G_2$ is a θ-connected graph where $\theta \le |G_1| + |G_2|$, equality only when G_1 and G_2 are mutually disjoint.

Finally, a certain description of 2-connected graphs will be used at a crucial enough moment in our work to merit being named. Let G be a 2-connected graph, $G = \{(q, r, s), (u, v, w)\}$, $(q, r, s), (u, v, w) \in [n]_e^3$. Let $t \in \{\sigma^2, \gamma, \rho, \tau, \nu, \eta, \xi_1, \xi_2\}$. Then, we will refer to G as of type t or a type t 2-connected graph iff $Cov(Y_{q,r,s}, Y_{u,v,w}) = E(Y_{q,r,s}Y_{u,v,w}) = t$.

We will omit specifying that a graph lives on $[n]_e^3$, when it is obvious by context, and for our work, all graphs are on $[n]_e^3$ for n an arbitrarily large member of $\mathbb{Z}^+$. Of course, note that if a graph lives on $[n]_e^3$, it lives on $[n+1]_e^3$. If it is obvious that we are restricting to points of a connected graph G when talking about communication, we will omit the dependence of the communication upon G.

Example The following is a picture of two mutually separated connected graphs $G_1 = \{(1, 2), (5, 1)\}$ and $G_2 = \{(2, 3), (3, 4), (4, 5)\}$. G_1 is a 2-connected graph, while G_2 is a

3-connected graph.



To proceed, we require some notation and results from the theory of unordered partitions.

2.3.2 Unordered Partitions of a Positive Integer k

Let $\kappa \in \mathbb{Z}^+$. Define $\mathcal{P}_{[\kappa]}$ as the collection of all unordered partitions of $\{1, \ldots, \kappa\}$, which may be thought of as all partitions of κ indistinguishable balls. Denote the magnitude of $\mathcal{P}_{[\kappa]}$ as $p(\kappa)$, i.e. $|\mathcal{P}_{[\kappa]}| = p(\kappa)$. Stanley documents many useful results regarding unordered partitions in [2], and it is well known that $p(\kappa) \leq 2^\kappa$, see for example Andrews, Chapter 5 [1]. We consider a partition $\sigma \in \mathcal{P}_{[\kappa]}$, to be a collection of positive integers that sum to κ. Each element of σ is called a part, and we will let the total number of parts of σ be denoted as $\mathcal{C}_\sigma$. Then, we enumerate the parts of σ as $r_1^\sigma, \ldots, r_{\mathcal{C}_\sigma}^\sigma$, giving $\sigma = \{r_s^\sigma\}_{s=1}^{\mathcal{C}_\sigma}$.

Example Let $\kappa = 4$: $\sigma_1 = 4$, $\sigma_2 = 3+1$, $\sigma_3 = 2+2$, $\sigma_4 = 2+1+1$, $\sigma_5 = 1+1+1+1$. So, $p(4) = 5$.

Using the dependence structure of $\{Y_{i,j,k} : (i,j,k) \in [n]_e^3\}$, we may partition $\mathbf{A}_\kappa^n$ by associating each element $\mathbf{A}_\kappa^n$ to exactly one element of $\mathcal{P}_{[\kappa]}$. For any $1 \leq r \leq |[n]_e^3|$, let

$$g_n(r) \;=\; \Big\{ G \subset \mathbf{A}_r^n : |G| = r,\ G \text{ connected} \Big\}$$

be the set of all r-connected graphs on $[n]_e^3$. For $\sigma \in \mathcal{P}_{[\kappa]}$ with parts $\{r_s^\sigma\}_{s=1}^{\mathcal{C}_\sigma}$, let

$$g_n^\kappa(\sigma) = \Big\{ G \in \mathbf{A}_\kappa^n : G = \bigcup_{s=1}^{\mathcal{C}_\sigma} G_s, G_s \in g_n(r_s^\sigma),\ s = 1, \ldots, \mathcal{C}_\sigma;\ G_1, \ldots, G_{\mathcal{C}_\sigma} \text{ mutually separated} \Big\}.$$

We observe that any κ-graph can be uniquely partitioned into a union of mutually separated, connected subgraphs. This fact is likely to be intuitively obvious; however, we prove it in Lemma 4.2 in Section 4 for completeness. Furthermore, on this basis of this fact, we have the following proposition.

2.3.3 Step 3 Proposition

Proposition 2.4

$$
\begin{aligned}
1 + \sum_{\kappa=1}^{|[n]_e^3|} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in \mathbf{A}_\kappa^n} & E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}} \\
&= 1 + \sum_{\kappa=1}^{|[n]_e^3|} \sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}}.
\end{aligned}
$$

Proof. See Section 4. **Q.E.D.**

Note that it is quite possible that $g_n^\kappa(\sigma)$ may be 0 for some $n, \ \kappa \in \mathbb{Z}^+$. For example, $m_i = 1$ for all $i \in \mathbb{Z}^+$, then when $n < \kappa \leq \frac{n^2}{2}$, there is no way to pick κ mutually separated 2-connected graphs. Thus, $g_n^{2\kappa}(\sigma)$ is zero for that σ comprising κ 2's.

2.4 Step 4:

Proposition 2.5 *Let $n, \ \kappa \in \mathbb{Z}^+$, $6 \leq \kappa \leq |[n]_e^3|$. There exists a function $f : \mathbb{Z}^+ \to \mathbb{R}^+$ such that*

$$
\sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^\kappa(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})| \frac{|t|^\kappa}{n^{(3\kappa)/2}} \leq f(\kappa),
$$

and

$$
\sum_{\kappa=6}^{\infty} f(\kappa) < \infty.
$$

Proof. See Section 4. **Q.E.D.**

Thus, by dominated convergence, we have established that

$$
\begin{aligned}
\lim_n \sum_{\kappa=1}^{|[n]_e^3|} \sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^\kappa(\sigma)} & E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}} \\
= & \lim_n \sum_{\kappa=1}^{5} \sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}} \\
& + \lim_n \sum_{\kappa=6}^{|[n]_e^3|} \sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\kappa=1}^{5} \lim_n \sum_{\sigma\in\mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^{\kappa}}{n^{(3\kappa)/2}} \\
&\quad + \sum_{\kappa=6}^{\infty} \lim_n \sum_{\sigma\in\mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^{\kappa}}{n^{(3\kappa)/2}} \\
&= \sum_{\kappa=1}^{\infty} \lim_n \sum_{\sigma\in\mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^{\kappa}}{n^{(3\kappa)/2}},
\end{aligned}
$$

dominated convergence being used in the second equality. Note that we must use dominated convergence as

$$
|[n]_e^3| = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} m_{i,j} = \sum_{i,j=1}^{n} m_i m_j - \sum_{i=1}^{n} m_i^2 = n^2(\alpha^2(1+o_n(1))),
$$

implying that $|[n]_e^3| \to \infty$ as $n \to \infty$.

2.5 Step 5:

We now find the limit of each of these summands. However, before doing such, we make the following convention:

Convention: Suppose that for a $k \in \mathbb{Z}^+$, we have a collection of constants $\{c_n\}_{n\geq k}$. We use the notation $c_n = o_n(g(n))$ to mean that as n ranges in $\mathbb{Z}^+\backslash[k]$, $\frac{c_n}{g(n)} \to 0$. This is the familiar "**little o of** $g(n)$" notation, and we are simply addressing the fact that c_n is not defined for $k > n$.

Proposition 2.6 *Let* $n,\ \kappa \in \mathbb{Z}^+$, $\kappa \leq |[n]_e^3|$.

$$
\begin{aligned}
&\sum_{\sigma\in\mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^{\kappa}}{n^{(3\kappa)/2}} \\
&\qquad = \begin{cases} \frac{1}{(\kappa/2)!}\left(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2}\right)^{\kappa/2} t^{\kappa}(1+o_n(1)) & : \ \kappa \text{ even} \\ t^{\kappa} o_n(1) & : \ \kappa \text{ odd.} \end{cases}
\end{aligned}
$$

Proof. See Section 4. **Q.E.D.**

2.6 Step 6:

We now complete the proof using Propositions 2.5 and 2.6.

Using the dominated convergence, whose validity is established in Proposition 2.5, and the limiting value of summands established in Proposition 2.6,

$$\begin{aligned}
&\lim_n 1+\sum_{\kappa=1}^{|[n]_e^3|}\sum_{\sigma\in\mathcal{P}_{[\kappa]}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}}\\
&= 1+\sum_{\kappa=1}^{\infty}\lim_n\sum_{\sigma\in\mathcal{P}_{[\kappa]}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}}\\
&= 1+\sum_{\kappa=1}^{\infty}\lim_n\Big(\frac{1}{(2\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{2\kappa/2}t^{2\kappa}(1+o_n(1))+t^{2\kappa-1}o_n(1)\Big)\\
&= 1+\sum_{\kappa=1}^{\infty}\lim_n\Big(\frac{1}{\kappa!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa}(t^2)^{\kappa}(1+o_n(1))+o_n(1)\Big)\\
&= \sum_{\kappa=0}^{\infty}\frac{(\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2})^\kappa}{\kappa!}=e^{\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.
\end{aligned}$$

In light of Remark 2.1, the proof has been completed. **Q.E.D.**

3 Proof of Theorem 1.2

We now prove a lemma to help us estimate the parameters of interest. Recall that

$$\begin{aligned}
\Omega_1 &= \{\omega\in\Omega:\lim_n\frac{1}{n}\sum_{i=1}^{n}M_i(\omega)=\alpha,\ \lim_n\frac{1}{n}\sum_{i=1}^{n}(M_i(\omega))^2=\beta,\\
&\qquad \lim_n\frac{1}{n}\sum_{i=1}^{n}(M_i(\omega))^j=E((M_1)^j)<\infty,\ 1\le j\le 4\}\\
L &= \{M_i(\Omega_1)\}_{i=1}^{\infty},
\end{aligned}$$

Let $T=\{t_1=\rho,t_2=\tau,t_3=\nu,t_4=\eta\}$, and

$$\begin{aligned}
\langle a(t_1),b(t_1)\rangle &= \langle 0,0\rangle,\\
\langle a(t_2),b(t_2)\rangle &= \langle 1,1\rangle,\\
\langle a(t_3),b(t_3)\rangle &= \langle 0,1\rangle,\\
\langle a(t_4),b(t_4)\rangle &= \langle 1,0\rangle\ .
\end{aligned}$$

Lemma 3.1 *Let $\{m_i\}_{i=1}^{\infty} \in L$. Then, for $1 \leq l \leq 4$,*

$$\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} \xrightarrow{P} \alpha^2 \beta t_l \quad .$$

Proof. See Section 5. **Q.E.D.**

Then, as it may be recalled $\mathbb{P}(\Omega_1) = 1$, it follows that for $1 \leq l \leq 4$, and $\epsilon > 0$, that

$$\begin{aligned}
& \mathbb{P}(|\hat{V}_{l,n} - \alpha^2 \beta t_l| > \epsilon) \\
= \ & E(I_{|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon}(\omega)) \\
= \ & E(1_{\Omega_1}(\omega) I_{|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon}(\omega)) \\
= \ & E(1_{\{\omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L\}}(\omega) I_{|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon}(\omega)) \\
= \ & E(1_{\{\omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L\}(\omega)} E(I_{|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon} | \{M_i\}_{i=1}^{\infty})(\omega)) \\
= \ & E(1_{\{\omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L\}}(\omega) \mathbb{P}(|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon | \{M_i\}_{i=1}^{\infty})(\omega)) \\
= \ & E(1_{\{\omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L\}} \\
& \mathbb{P}(|\frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{s_1=1}^{M_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{M_{h_{b(t_l)}(g_{a(t_l)}(i,j),k)}} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2} - \alpha^2\beta t_l| > \epsilon | \{M_i\}_{i=1}^{\infty})(\omega)) \\
= \ & E^{\{M_i\}_{i=1}^{\infty}}(1_L(\{m_i\}_{i=1}^{\infty}) \\
& \mathbb{P}(|\frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{s_1=1}^{m_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(i,j),k)}} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2} - \alpha^2\beta t_l| > \epsilon | \{M_i\}_{i=1}^{\infty} = \{m_i\}_{i=1}^{\infty})) \\
= \ & E^{\{M_i\}_{i=1}^{\infty}}(1_L(\{m_i\}_{i=1}^{\infty}) \\
& \mathbb{P}(|\frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{s_1=1}^{m_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(i,j),k)}} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2} - \alpha^2\beta t_l| > \epsilon)).
\end{aligned}$$

As Lemma 3.1 implies that for all $\{m_i\}_{i=1}^{\infty} \in L$,

$$\mathbb{P}(|\frac{1}{n^3} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{s_1=1}^{m_{i,j}} Y_{i,j,s_2} \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n} \sum_{s_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(i,j),k)}} Y_{h_{b(t_l)}(g_{a(t_l)}(i,j),k),s_2} - \alpha^2\beta t_l| > \epsilon) \ = \ o_n(1),$$

then it follows by dominate convergence that

$$\mathbb{P}(|\hat{V}_{l,n} - \alpha^2\beta t_l| > \epsilon) \ = \ o_n(1),$$

or equivalently that $\hat{V}_{l,n} \xrightarrow{P} \alpha^2\beta t_l$. It then follows by Slutsky's Theorem that $\sum_{i=1}^4 V_{i,n} \xrightarrow{P} \alpha^2\beta(\rho+\tau+\nu+\eta)$. Further, we also now that since $f(x)=x^{-1/2}$ is continuous at $\alpha^2\beta(\rho+\tau+\nu+\eta)>0$, implying $(\sum_{i=1}^4 V_{i,n})^{-1/2} \xrightarrow{P} (\alpha^2\beta(\rho+\tau+\nu+\eta))^{-1/2}$. Then, given Theorem 1.1, a simple application of Slutsky's theorem gives

$$\frac{1}{\sqrt{\sum_{i=1}^4 \hat{V}_{i,n}}} \frac{1}{n^{3/2}} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sum_{k=0}^{M_{i,j}} Y_{i,j,k} \xrightarrow{d} \mathrm{N}(0,1) \quad .$$

Q.E.D.

4 Proof of Propositions for Theorem 1.1

4.1 Proof of Proposition 2.1

Proposition 2.1 *Let $t \in \mathbb{R}$. If*

$$E\exp\{t\frac{1}{n^{3/2}} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \to e^{\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}$$

for all $\{m_i\}_{i=1}^\infty \in \{M_i(\Omega_1)\}_{i=1}^\infty$, then

$$E\exp\{it\frac{1}{n^{3/2}} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sum_{k=1}^{M_{i,j}} Y_{i,j,k}\} \to e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.$$

Proof. Recall that

$$\begin{aligned}\Omega_1 &= \{\omega\in\Omega : \lim_n \frac{1}{n}\sum_{i=1}^n M_i = \alpha,\ \lim_n \frac{1}{n}\sum_{i=1}^n (M_i)^2 = \beta, \\ &\qquad \lim_n \frac{1}{n}\sum_{i=1}^n (M_i)^j = E((M_1)^j) < \infty,\ 1\le j\le 4\}.\end{aligned}$$

Then, if we define

$$\begin{aligned}L &= \{\{m_i\}_{i=1}^\infty \in (\mathbb{N})^{\mathbb{Z}^+} : \lim_n \frac{1}{n}\sum_{i=1}^n m_i = \alpha,\ \lim_n \sum_{i=1}^n (m_i)^2 = \beta, \\ &\qquad \lim_n \frac{1}{n}\sum_{i=1}^n (m_i)^j = E((M_1)^j) < \infty,\ 1\le j\le 4\},\end{aligned}$$

it follows that,

$$\Omega_1 \;=\; \{\omega \in \Omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L\}.$$

Thus, $\Omega_1 \in \sigma(\{M_i(\omega)\}_{i=1}^{\infty})$. In addition, by the ergodic the Ergodic Theorem (Law of Large Numbers), we have that

$$\mathbb{P}(\omega \in \Omega : \{M_i(\omega)\}_{i=1}^{\infty} \in L) = \mathbb{P}(\Omega_1) = 1.$$

Now, letting $i = \sqrt{-1}$, it follows that

$$\begin{aligned}
E \exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}} Y_{i,j,k}\} &= E\Big(1_{\Omega_1}(\omega)\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}(\omega)} Y_{i,j,k}(\omega)\}\Big)\\
&= E\Big(E(1_{\Omega_1}(\omega)\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}} Y_{i,j,k}\}|\{M_i\}_{i=1}^{\infty})(\omega)\Big)\\
&= E\Big(1_{\Omega_1}(\omega)E(\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}} Y_{i,j,k}\}|\{M_i\}_{i=1}^{\infty})(\omega)\Big)\\
&= E^{\{M_i\}_{i=1}^{\infty}}\Big(1_L(\{m_i\}_{i=1}^{\infty})E(\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\}|\{M_i\}_{i=1}^{\infty} = \{m_i\}_{i=1}^{\infty})\Big)\\
&= E^{\{M_i\}_{i=1}^{\infty}}\Big(1_L(\{m_i\}_{i=1}^{\infty})E(\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\})\Big)
\end{aligned}$$

Now, because

$$E \exp\{t\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \to e^{\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}$$

if and only if

$$E \exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \to e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}},$$

it follows by our assumption and the bounded convergence theorem that

$$\lim_{n} E^{\{M_i\}_{i=1}^{\infty}}\Big(1_L(\{m_i\}_{i=1}^{\infty})E(\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}} Y_{i,j,k}\})\Big)$$

$$\begin{aligned}
&\to E^{\{M_i\}_{i=1}^\infty}(1_L(\{m_i\}_{i=1}^\infty)e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}})\\
&= e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}}.
\end{aligned}$$

Thus, it follows that

$$E\exp\{it\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{M_{i,j}}Y_{i,j,k}\}\to e^{-\alpha^2\beta(\rho+\tau+\nu+\eta)\frac{t^2}{2}} \quad .$$

Q.E.D.

Recall that

$$\begin{aligned}
\Omega_1 &= \{\omega\in\Omega : \lim_n \frac{1}{n}\sum_{i=1}^n M_i(\omega)=\alpha,\ \lim_n\frac{1}{n}\sum_{i=1}^n(M_i(\omega))^2=\beta,\\
&\lim_n\frac{1}{n}\sum_{i=1}^n(M_i(\omega))^j = E((M_1)^j)<\infty,\ 1\le j\le 4\}.
\end{aligned}$$

Because it is assumed that $\{m_i\}_{i=1}^\infty\in\{M_i(\Omega_1)\}_{i=1}^\infty$, if follows that

$$\begin{aligned}
&\lim_n\frac{1}{n}\sum_{i=1}^n m_i=\alpha,\ \lim_n\sum_{i=1}^n(m_i)^2=\beta,\\
&\lim_n\sum_{i=1}^n(m_i)^j=E((M_1)^j)<\infty,\ 1\le j\le 4. \qquad (1)
\end{aligned}$$

Here, we introduce another preliminary lemma to be used later that describes some salient behaviour of the $\{m_i\}_{i=1}^\infty\in\{M_i(\Omega_1)\}_{i=1}^\infty$. Let $g_0,g_1:\mathbb{N}^2\to\mathbb{N}$ such that for $(x,y)\in\mathbb{N}^2$, $g_0(x,y)=x,\ g_1(x,y)=y$. Let $h_0,h_1:\mathbb{N}^2\to\mathbb{N}^2$ such that for $(x,y)\in\mathbb{N}^2$, $h_0(x,y)=(x,y)$, $h_1(x,y)=(y,x)$.

Lemma 4.1 *For $\langle a_1,b_1\rangle,\langle a_2,b_2\rangle,\langle a_3,b_3\rangle\in\{0,1\}^2$, let*

$$\begin{aligned}
\lambda_n^1 &= \frac{1}{n^2}\sum_{s_1,s_2=1}^n m_{s_1,s_2}\\
\lambda_n^2(a_1,b_1) &= \frac{1}{n^3}\sum_{s_1,s_2=1}^n m_{s_1,s_2}\sum_{s_3=1}^n m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\\
\lambda_n^3(a_1,b_2;a_2,b_2) &= \frac{1}{n^4}\sum_{s_1,s_2=1}^n m_{s_1,s_2}\sum_{s_3=1}^n m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\sum_{s_4=1}^n m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)}
\end{aligned}$$

$$\lambda_n^4(a_1,b_2;a_2,b_2;a_3,b_3) = \frac{1}{n^5}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\sum_{s_4=1}^{n} m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)} \times \sum_{s_5=1}^{n} m_{h_{b_3}(g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)),s_5)} .$$

Then, there exists a constant θ such that for all $\langle a_1,b_1\rangle,\ldots,\langle a_3,b_3\rangle \in \{0,1\}^2$ and all $2 \le i \le 4$,

$$\lim_n \lambda_n^1 \le \theta$$
$$\lim_n \lambda_n^i(a_1,b_1;\cdots;a_{i-1},b_{i-1}) \le \theta .$$

Proof.

It suffices to show that $\lim_n \lambda_n^1$ and $\lim_n \lambda_n^i(a_1,b_1;\cdots;a_{i-1},b_{i-1})$ exists for $2 \le i \le 4$. We now proceed to this task.

As $\lambda_n^1 = \frac{1}{n^2}\sum_{s_1,s_2=1}^n m_{s_1,s_2} = (\frac{1}{n}\sum_{i=1}^n m_i)^2$, we have by assumption that $\lim_n \lambda_n^1 = \{(EX_1)^2\}$.

For any $\langle a_1,b_1\rangle,\ldots,\langle a_{i-1},b_{i-1}\rangle \in \{0,1\}^2$, and $1 \le s_1,\ldots,s_5 \le n$, because $m_{i,j} = m_i m_j$ for $i,j \in \mathbb{Z}^+$, we have that

$$\begin{aligned} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)} &= m_{g_{a_1}(s_1,s_2),s_3} \\ m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)} &= m_{g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4} \\ m_{h_{b_3}(g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)),s_5)} &= m_{g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)),s_5} , \end{aligned}$$

and

$$\begin{aligned} g_{a_1}(s_1,s_2) &\in \{1,2\}, \\ g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)) &\in \{1,\ldots,3\}, \\ g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)) &\in \{1,\ldots,4\} . \end{aligned}$$

Thus,

$$\begin{aligned} \lim_n \frac{1}{n^3}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)} &= \lim_n \frac{1}{n^3}\sum_{s_1=1}^{n} m_{s_1}^{1+i_1}\sum_{s_2=1}^{n} m_{s_2}^{1+i_2}\sum_{s_3=1}^{n} m_{s_3} \\ &= EX_1^{1+i_1}EX_1^{1+i_2}EX_1 \end{aligned}$$

for some $i_1,i_2 \in \{0,1\}$, $i_1+i_2=1$. Similarly,

$$\lim_n \frac{1}{n^4}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\sum_{s_4=1}^{n} m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)}$$

$$
\begin{aligned}
&= \lim_n \frac{1}{n^4} \sum_{s_1=1}^{n} m_{s_1}^{1+i_1+j_1} \sum_{s_2=1} m_{s_2}^{1+i_2+j_2} \sum_{s_3=1}^{n} m_{s_3}^{1+j_3} \sum_{s_4=1}^{n} m_{s_4} \\
&= EX_1^{1+i_1+j_1} EX_1^{1+i_2+j_2} EX_1^{1+j_3} EX_1
\end{aligned}
$$

for some $i_1, i_2, j_1, \ldots, j_3 \in \{0, 1\}$, $i_1 + i_2 = j_1 + \cdots + j_3 = 1$, and

$$
\lim_n \frac{1}{n^5} \sum_{s_1,s_2=1}^{n} m_{s_1,s_2} \sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)} \sum_{s_4=1}^{n} m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)} \times
$$

$$
\sum_{s_5=1}^{n} m_{h_{b_3}(g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)),s_5)}
$$

$$
\begin{aligned}
&= \lim_n \frac{1}{n^5} \sum_{s_1=1}^{n} m_{s_1}^{1+i_1+j_1+k_1} \sum_{s_2=1} m_{s_2}^{1+i_2+j_2+k_2} \sum_{s_3=1}^{n} m_{s_3}^{1+j_3+k_3} \sum_{s_4=1}^{n} m_{s_4}^{1+k_4} \sum_{s_5=1}^{n} m_{s_5} \\
&= EX_1^{1+i_1+j_1+k_1} EX_1^{1+i_2+j_2+k_2} EX_1^{1+j_3+k_3} EX_1^{1+k_4} EX_1
\end{aligned}
$$

for some $i_1, i_2, j_1, \ldots, j_3, k_1, \ldots, k_4 \in \{0, 1\}$, $i_1 + i_2 = j_1 + \cdots + j_3 = k_1 + \cdots + k_4 = 1$. **Q.E.D.**

4.2 Proof of Proposition 2.2

Proposition 2.2

$$
\begin{aligned}
E\{\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \prod_{k=1}^{m_{i,j}} (1 + \frac{1}{n^{3/2}} t Y_{i,j,k})\} &\leq E \exp\{t \frac{1}{n^{3/2}} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{k=1}^{m_{i,j}} Y_{i,j,k}\} \\
&\leq (1 + o_n(1)) E\{\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \prod_{k=1}^{m_{i,j}} (1 + \frac{1}{n^{3/2}} t Y_{i,j,k})\}.
\end{aligned}
$$

Proof. Let $1 \leq i, j \leq n$, $1 \leq k \leq m_{i,j}$. From Taylor's Theorem,

$$
1 + t\frac{1}{n^{3/2}} Y_{i,j,k} \leq 1 + t\frac{1}{n^{3/2}} Y_{i,j,k} + \frac{C(Y_{i,j,k}, t)}{2} t^2 \frac{1}{n^3} Y_{i,j,k}^2 = \exp\{t\frac{1}{n^{3/2}} Y_{i,j,k}\},
$$

where $C(Y_{i,j,k}, t)$ is a random variable such that $0 < C(Y_{i,j,k}, t) < e^{n^{-3/2} t |Y_{i,j,k}|} < e^{tC}$ for all i, j. Then, for n large enough such that $0 < 1 - \frac{Ct}{n^{3/2}}$, and $\frac{tC}{n^{3/2}} < 1$,

$$
E\{\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \prod_{k=1}^{m_{i,j}} (1 + t\frac{1}{n^{3/2}} Y_{i,j,k})\}
$$

$$
\begin{aligned}
&\le E\exp\{t\frac{1}{n^{3/2}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}Y_{i,j,k}\} \\
&= E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k}+\frac{C(Y_{i,j,k},t)}{2}t^2\frac{1}{n^3}Y_{i,j,k}^2)\} \\
&= E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\frac{(1+t\frac{1}{n^{3/2}}Y_{i,j,k}+\frac{C(Y_{i,j,k},t)}{2}t^2\frac{1}{n^3}Y_{i,j,k}^2)}{(1+t\frac{1}{n^{3/2}}Y_{i,j,k})}\} \\
&\le E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})(1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})\} \\
&= \prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\} \\
&= (1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})^{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}}E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\} \\
&= (1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})^{n^2(\frac{1}{n}^2\sum_{\substack{i,j=1\\ i\neq j}}^{n}m_{i,j})}E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\} \\
&= (1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})^{n^2((\frac{1}{n}\sum_{i=1}^{n}m_i)^2-\frac{1}{n}^2\sum_{i=1}^{n}(m_i)^2)}E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\}
\end{aligned}
$$

The second inequality follows because $1+t\frac{1}{n^{3/2}}Y_{i,j,k}>0$, as we have assumed $\frac{tC}{n^{3/2}}<1$.

By equation (1), it follows that

$$
(\frac{1}{n}\sum_{i=1}^{n}m_i)^2-\frac{1}{n}^2\sum_{i=1}^{n}(m_i)^2=\alpha^2(1+o_n(1)).
$$

Thus,

$$
\begin{aligned}
&(1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})^{n^2((\frac{1}{n}\sum_{i=1}^{n}m_i)^2-\frac{1}{n}^2\sum_{i=1}^{n}(m_i)^2)}E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\} \\
&\quad= (1+\frac{\frac{e^{tC}(tC)^2}{2n^3}}{1-\frac{Ct}{n^{3/2}}})^{n^2\alpha^2(1+o_n(1))}E\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\prod_{k=1}^{m_{i,j}}(1+t\frac{1}{n^{3/2}}Y_{i,j,k})\}
\end{aligned} \tag{2}
$$

Clearly,

$$\Big(1+\frac{1}{n^2}\Big(\frac{\frac{e^{tC}(tC)^2}{2n}}{1-\frac{Ct}{n^{3/2}}}\Big)\Big)^{n^2\alpha^2(1+o_n(1))} \quad = \quad 1+o_n(1),$$

which completes the proof. **Q.E.D.**

4.3 Proof of Proposition 2.4

We need 3 preliminary lemmas before proving Proposition 2.4.

4.3.1 Preliminary Lemmas

Lemma 4.2 *For any κ-graph G on $[n]_e^3$, there exists an $N(G) \in [\kappa]$ and graphs $\{B(G)_s\}_{s=1}^{N(G)}$ such that the $B(G)_s$ are mutually separated graphs on $[n]_e^3$, each is a connected graph, and $\bigcup_{s=1}^{N(G)} B(G)_s = G$.*

Proof. This follows just by recognizing the two following facts. If $y \in G$ and $G(y)$ is the graph comprising those points $x \in G \subset [n]_e^3$ such that x and y communicate in G, then $G(y)$ is a connected graph. Furthermore, $G(y)$ and $G\backslash G(y)$ are mutually separated graphs. Thus, we choose a point $x_1 \in G$, and let $B(G)_1$ to be the collection of all points that communicate with x_1 in G. We then continue this process with $G\backslash B(G)_1$, and the process ends with the given partition of G. **Q.E.D.**

Lemma 4.3 *Let $\kappa \in \mathbb{Z}^+$, and $G = \{(i_s, j_s, k_s)\}_{s=1}^{\kappa}$ be a κ-graph on $[n]_e^3$. Then,*

$$E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \quad = \quad \prod_{v=1}^{N(G)} E\Big(\prod_{(w,y,z)\in B(G)_v} Y_{w,y,z}\Big)$$

Proof. This follows by definition, since all the $B(G)_1, \ldots, B(G)_{N(G)}$ are mutually separated, and thus, as alluded to earlier, $\{Y_{(i,j,k)}\}_{(i,j,k)\in B(G)_1}, \ldots, \{Y_{(i,j,k)}\}_{(i,j,k)\in B(G)_{N(G)}}$ are mutually independent collections of random variables. **Q.E.D.**

Recall the following notation. For any $1 \le r \le |[n]_e^3|$, let $g_n(r)$ be the set of all r-connected graphs on $[n]_e^3$. For $\sigma \in \mathcal{P}_{[\kappa]}$ with parts $\{r_s^\sigma\}_{s=1}^{\mathcal{C}_\sigma}$, let

$$g_n^\kappa(\sigma) = \Big\{G \in \mathbf{A}_\kappa^n : G = \bigcup_{s=1}^{\mathcal{C}_\sigma} G_s, G_s \in g_n(r_s^\sigma),\ s = 1, \ldots, \mathcal{C}_\sigma;\ G_1, \ldots, G_{\mathcal{C}_\sigma} \text{ mutually separated}\Big\}.$$

Lemma 4.4 *Let $\kappa \in \mathbb{Z}^+$. If $\sigma_1,\ \sigma_2 \in \mathcal{P}_{[\kappa]}$, $\sigma_1 \neq \sigma_2$, then $g_n^\kappa(\sigma_1) \bigcap g_n^\kappa(\sigma_2) = \emptyset$, and*

$$\mathbf{A}_\kappa^n = \bigcup_{\sigma \in \mathcal{P}_{[\kappa]}} g_n^\kappa(\sigma). \tag{3}$$

Proof. Now, if G is a κ-graph on $[n]_e^3$, then in reference to Lemma 4.2, $\sum_{s=1}^{N(G)} |B(G)_s| = \kappa$, and so $G \in g_n^\kappa(\{|B(G)_s|\}_{s=1}^{N(G)})$, which implies equation (3). Similarly, if $G \in g_n^\kappa(\sigma_1)$ and $G \in g_n^k(\sigma_2)$, then $\sigma_1 = \{|B(G)_s|\}_{s=1}^{N(G)} = \sigma_2$, which is a contradiction. **Q.E.D.**

4.3.2 Proof of Proposition

Proposition 2.4

$$\begin{aligned} 1 + \sum_{\kappa=1}^{|[n]_e^3|} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in \mathbf{A}_\kappa^n} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}} \\ = 1 + \sum_{\kappa=1}^{|[n]_e^3|} \sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}}. \end{aligned}$$

Proof. Lemma 4.4 completes the proof of the proposition. **Q.E.D.**

4.4 Proof of Propositions 2.5

We need 6 preliminary lemmas and three corollaries before proving Proposition 2.5, and these lemmas require the introduction of some new notation.

For distinct points $(q,r,1),(u,v,1) \in G \subset [n]_1^3$, we say that $(q,r,1)$ and $(u,v,1)$ are horizontal G-relations if

1. $u = q$ and $(q,l,1) \notin G,\ \min\{r,v\} < l < \max\{r,v\}$.

For distinct points $(q,r,1),(u,v,1) \in G \subset [n]_1^3$, we say that $(q,r,1)$ and $(u,v,1)$ are vertical G-relations if

2. $v = r$ and $(l,r,1) \notin G,\ \min\{q,u\} < l < \max\{q,u\}$.

We will say that y is a horizontal, respectively vertical, G-relation of x if x and y are horizontal, respectively vertical, G-relations. We will say that y is a G-relation of x if x and y are either vertical G-relations or horizontal G-relations.

For distinct points $x = (i,j,1),\ y = (q,r,1) \in G \subset [n]_1^3$, we say that $y \in G$ is an h/v G-neighbor of x if one of the 3 following relationships hold:

1. $(r,q)=(i,j)$.
2. y is a horizontal G-relation of x.
3. y is a vertical G-relation of $(j,i,1)$.

Similarly, for distinct points $x=(i,j,1),\ y=(q,r,1)\in G\subset[n]^3_1$, we say that $y\in G$ is a v/h G-neighbor of x if one of the 3 following relationships hold:

1. $(r,q)=(i,j)$.
2. y is a vertical G-relation of x.
3. y is a horizontal G-relation of $(j,i,1)$.

We define the G-neighbors of x as the collection of all points that are either a v/h G-neighbor of x or a h/v G-neighbor of x.

4.4.1 Preliminary Lemmas

Recall that

$$[n]^3_1 \;=\; \{(i,j,k)\in[n]^3_e:\ k=1\}.$$

Recall that for a graph $G\in\mathbb{Z}^3_{e,+}$, we define the projection of G as $Proj(G)$ where

$$Proj(G) \;=\; \{(i,j,1)\in\mathbb{Z}^3_{e,+}:\ (i,j,k)\in G \text{ for some } 1\le k\le m_{i,j}\}.$$

In addition, for any $G\in\mathbb{Z}^3_{1,+},\ H\in\mathbb{Z}^3_{e,+}$, we define the inverse projection of G onto H as $Proj^{-1}_G(H)$ where

$$Proj^{-1}_G(H) \;=\; \{(i,j,k)\in H:\ (i,j,1)\in G\}.$$

Lemma 4.5 *For any $G\in[n]^3_e$, $Proj(G)\in[n]^3_1$, and $G\in[n]^3_e$ is connected iff $Proj(G)$ is connected. Furthermore, if $E\subset Proj(G)$ and E is connected, then $Proj^{-1}_E(G)$ is connected.*

Proof. That $Proj(G)\in[n]^3_1$ for any $G\in[n]^3_e$, and that $G\in[n]^3_e$ is connected iff $Proj(G)$ is connected are statements that follow directly from the definition of a connected graph. Finally, if $E\subset Proj(G)$ and E is connected, then $Proj^{-1}_E(G)$ is connected also by construction. **Q.E.D.**

Lemma 4.6 *Let G be a connected graph on $[n]^3_1$, $|G|>1$, and let x be in G. Then, $G\backslash\{x\}=\overline{G}(x)\cup\underline{G}(x)$ where $\overline{G}(x)$ and $\underline{G}(x)$ are connected subgraphs of G that are either mutually separated or they touch. Further, we may take $\overline{G}(x)$ to be all individuals of G that communicate in $G\backslash\{x\}$ with the h/v G-neighbors of x, and $\underline{G}(x)$ to be all individuals of G that communicate in $G\backslash\{x\}$ with the v/h G-neighbors of x.*

Note, when $\overline{G}(x)$ and $\underline{G}(x)$ touch, $G\backslash\{x\}$ is a connected set. Also, it may happen that either $\overline{G}(x) = \emptyset$ or $\underline{G}(x) = \emptyset$, but they cannot both be empty. Since if either one of them is empty, we have by convention that they are connected, it follows that both must be non-empty if they are mutually separated.

Proof. Let $\overline{G}(x)$ be all individuals of G that communicate in $G\backslash\{x\}$ with the h/v G-neighbors of x. Let $\underline{G}(x)$ be all individuals of G that communicate in $G\backslash\{x\}$ with the v/h G-neighbors of x. If $y \in G$, then since G is connected, there is a sequence of distinct points $z_1, \ldots, z_l \in G$, $1 \leq l \leq n-2$, such that

$$y \leftrightarrow z_1 \cdots \leftrightarrow z_l \leftrightarrow x.$$

But then by construction, z_l is a G-neighbor and y communicates in $G\backslash\{x\}$ with z_l. Thus, each $y \in G$ communicates in $G\backslash\{x\}$ with either the h/v G-neighbors of x or with the v/h G-neighbors of x.

We now conclude that $G\backslash\{x\} = \overline{G}(x) \cup \underline{G}(x)$. Since all h/v G-neighbors touch, and all v/h G-neighbors touch, by construction, both $\overline{G}(x)$ and $\underline{G}(x)$ are connected subgraphs of G. They are either mutually separated or not, which proves the lemma. **Q.E.D.**

Corollary 4.1 *Let G be a connected graph on $[n]_e^3$, $|G| > 1$, and let $\mathbf{x}$ be the cylinder defined as $\mathbf{x} = \{(i,j,k) \in G : i = i',\ j = j',\ 1 \leq k \leq m_{i',j'}\}$ for some $(i',j',k') \in G$. Then, $G\backslash\{\mathbf{x}\} = \overline{G}(\mathbf{x}) \cup \underline{G}(\mathbf{x})$ where $\overline{G}(\mathbf{x})$ and $\underline{G}(\mathbf{x})$ are connected subgraphs of G that are either mutually separated or they touch.*

Proof. From Lemma 4.5, $Proj(G)$ is connected. $Proj(\mathbf{x})$ is a point of $Proj(G)$. Thus, by Lemma 4.6, we have that

$$Proj(G)\backslash\{Proj(\mathbf{x})\} = H_1 \cup H_2,$$

where H_1 and H_2 are disjoint, connected subgraphs of $Proj(G)$ on $[n]_1^3$. By Lemma 4.5, it follows that $Proj_{H_1}^{-1}(G)$ and $Proj_{H_2}^{-1}(G)$ are disjoint, connected subgraphs of G on $[n]_e^3$ such that

$$Proj_{H_1}^{-1}(G) \cup Proj_{H_2}^{-1}(G) \cup \mathbf{x} = G.$$

The proof is completed by noting that either $Proj_{H_1}^{-1}(G)$ and $Proj_{H_2}^{-1}(G)$ touch or they don't. **Q.E.D.**

Lemma 4.7 *Let G be an κ-connected graph on $[n]_e^3$, $2 \leq \kappa \leq |[n]_e^3|$. Then, there exist an $\kappa - 1$-connected graph, $H(G)$, and 1-connected graph, $H(G)_1$, i.e. a singleton, such that $H(G) \cup H(G)_1 = G$.*

Note, we will use $H(G)_1$ to refer to both the set and the singleton.

Proof. The proof is broken into two parts. We prove it first for $G \in [n]^3_1$, and then more generally for $G \in [n]^3_e$.

Case $G \in [n]^3_1$**:** The proof is by induction. It is clearly true for all 2-connected graphs. Suppose it is true for all i-connected graphs, $1 \le i \le \kappa - 1$.

Choose any point $x \in G$. By Lemma 4.6, $G \backslash \{x\}$ the union of at most 2 mutually separated, connected subgraphs of G, call them G_1 and G_2. Assume that G_1 is non-empty, acknowledging that G_2 may be empty, so that G_1 may equal $G\backslash\{x\}$. By a repeated use of the induction assumption, there exists a sequence of points $\{x_1, \ldots, x_{|G_1|}, x_{|G_1|+1}\}$ such that $x_{|G_1|+1} = x$, for all $2 \le l \le |G_1| + 1$, $\{x_s\}_{s=1}^{l-1}$ is a connected graph, $\{x_s\}_{s=1}^{l-1}$ and $\{x_l\}$ touch, and $G_1 = \{x_s\}_{s=1}^{|G_1|}$. If

$$x_1 \leftrightarrow \cdots \leftrightarrow x_{|G_1|+1},$$

then $\{x_s\}_{s=2}^{|G_1|} \cup \{x\} \cup G_2 = G\backslash\{x_1\}$ is a connected graph. Clearly, we can let $H(G) = G\backslash\{x_1\}$ and $H(G)_1 = x_1$. Otherwise, let

$$v = \max\{s : 1 \le s \le |G_1| \text{ such that } x_s \not\leftrightarrow x_{s+1}\}.$$

Then, since $\{x_{v+1}\}$ touches $\{x_s\}_{s=1}^{v}$, but $x_v \not\leftrightarrow x_{v+1}$, it follows that $\{x_s\}_{\substack{s=1\\ s\neq v}}^{v+1}$ is a connected graph. But, because

$$x_{v+1} \leftrightarrow \cdots \leftrightarrow x_{|G_1|+1} = x,$$

it follows that $\{x_s\}_{\substack{s=1\\ s\neq v}}^{|G_1|} \cup \{x\} \cup G_2 = G\backslash\{x_v\}$ is a connected graph. Thus, we can let $H(G) = G\backslash\{x_v\}$ and $H(G)_1 = x_v$.

Case $G \in [n]^3_e$**:** From Lemma 4.5, we have that $G \in [n]^3_e$ is connected iff $Proj(G)$ is connected. By, what we have proved, we can partition $Proj(G)$ into $Proj(G) = H(Proj(G)) \cup H(Proj(G))_1$, where $H(Proj(G))$ is a connected graph of magnitude less than or equal to $\kappa - 1$, $H(Proj(G))_1$ is a 1-connected graph. Now, by Lemma 4.5, it follows that both $Proj^{-1}_{H(Proj(G))}(G)$ are connected and $Proj^{-1}_{H(Proj(G))_1}(G)$ are connected. However, by construction, $Proj^{-1}_{H(Proj(G))_1}(G) \subset \{(i,j,k)\}_{k=1}^{m_{i,j}}$ for some $1 \le i \ne j \le n$, $1 \le m_{i,j}$. If for $1 \le \tilde{m}_{i,j} \le m_{i,j}$, we enumerate $Proj^{-1}_{H(G)_1}(G)$ as $\{(i,j,k_s)\}_{s=1}^{\tilde{m}_{i,j}}$, then our proof follows by letting $H(G) = Proj^{-1}_{H(Proj(G))}(G) \cup \{(i,j,k_s)\}_{s=1}^{\tilde{m}_{i,j}-1}$ and $H(G)_1 = (i,j,k_{\tilde{m}_{i,j}})$. Just note that some member of (q,r,s) in $Proj^{-1}_{H(Proj(G))}(G)$ is such that $a = b$ for some $a \in \{q,r\}$ and $b \in \{i,j\}$ by construction of $H(Proj(G)) \cup H(Proj(G))_1$. **Q.E.D.**

Lemma 4.8 *Let G be a 2κ-connected graph, $2 \le 2\kappa \le |[n]^3_e|$. Then, there exists κ mutually disjoint,* 2*-connected graphs, $H^2_i(G)$, $i = 1, \ldots, \kappa$, such that $\cup_{i=1}^{\kappa} H^2_i(G) = G$.*

Note that even though the κ 2-connected graphs are mutually disjoint, they are not mutually separated.

Proof. The proof is broken into two parts. We prove it first for $G \in [n]_1^3$, and then more generally for $G \in [n]_e^3$.

Case $G \in [n]_1^3$: This is an induction argument. The result clearly holds for the case $\kappa = 1$. Assume that the result holds for all $2l$-connected graphs, $2 \leq l < \kappa$.

Choose $x \in G$. By Lemma 4.6, $G \backslash \{x\} = \overline{G}(x) \cup \underline{G}(x)$, where $\overline{G}(x)$ is the collection of all individuals of G that communicate in $G \backslash \{x\}$ with the h/v G-neighbors of x, and $\underline{G}(x)$ is the collection of all individuals of G that communicate in $G \backslash \{x\}$ with the v/h G-neighbors of x. We then have that either $\overline{G}(x)$ and $\underline{G}(x)$ are mutually separated, or they are not. We now address each of the possible cases.

$\overline{G}(x)$ and $\underline{G}(x)$ are mutually separated: Assume $\overline{G}(x)$ and $\underline{G}(x)$ are mutually separated. By construction, they both are non-empty. If both $|\overline{G}(x)|$ and $|\underline{G}(x)|$ are odd, then it follows that $|x| + |\overline{G}(x)| + |\underline{G}(x)|$ is odd, which is a contradiction. Without loss of generality, assume that $|\overline{G}(x)|$ is even. Then, $x \cup \underline{G}(x)$ and $\overline{G}(x)$ are both connected graphs. Further, they are mutually disjoint, and their magnitudes are even and strictly less than 2κ. Thus, by applying our induction assumption independently for $x \cup \underline{G}(x)$ and $\overline{G}(x)$ gives the result.

$\overline{G}(x)$ and $\underline{G}(x)$ are not mutually separated: Assume that $\overline{G}(x)$ and $\underline{G}(x)$ are not mutually separated, which implies that they touch. If $\overline{G}(x) \neq \emptyset$, let y be an h/v G-neighbor of x. Otherwise, let y be a v/h G-neighbor of x. Then, $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ is either 1 or 2 mutually separated, connected subgraphs being formed. If $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ is connected, then because its magnitude is even and strictly less than 2κ, the induction hypothesis applied to $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ and $\{x, y\}$ completes the proof.

If $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ is not a connected set, then by Lemma 4.6, it follows that $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\} = \overline{H} \cup \underline{H}$, where $\underline{H}$ and $\overline{H}$ are non-empty, mutually separated, connected subgraphs of G, whose magnitudes are both either even or odd. $\overline{H}$ is all individuals of $\overline{G}(x) \cup \underline{G}(x)$ that communicate in $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ with h/v $\overline{G}(x) \cup \underline{G}(x)$-neighbors of y. $\underline{H}$ is all individuals of $\overline{G}(x) \cup \underline{G}(x)$ that communicate in $\overline{G}(x) \cup \underline{G}(x) \setminus \{y\}$ with v/h $\overline{G}(x) \cup \underline{G}(x)$-neighbors of y. Further, $|\underline{H}|$, $|\overline{H}| < 2\kappa$. If $|\underline{H}|$ and $|\overline{H}|$ are both even, then the induction hypothesis applied to $\underline{H}$, $\overline{H}$, and $\{x, y\}$ completes the proof.

If they are both odd, more work needs to be done. Let $x = (x_0, x_1, 1)$, $y_1 = (y_0, y_1, 1)$. Then, by construction, it follows that $x_{i^*} = y_{j^*}$ for some $i^* \in \{0, 1\}$ and some $j^* \in \{0, 1\}$. Then, by construction, since $\overline{H}$ is non-empty it contains some h/v $\overline{G}(x) \cup \underline{G}(x)$-neighbors of y. Thus, there exists a point $w = (w_0, w_1, 1) \in \overline{H}$ such that either $w_0 = y_0$ or $w_1 = y_0$. Similarly, by construction, since $\underline{H}$ is non-empty it contains some v/h $\overline{G}(x) \cup \underline{G}(x)$-neighbors of y. Thus, there exists a point $z = (z_0, z_1, 1) \in \overline{H}$ such that either $z_0 = y_1$ or $z_1 = y_1$. Thus, since $x_{i^*} = y_{j^*}$, it follows that either $x_{i^*} = w_0$, $x_{i^*} = w_1$, $x_{i^*} = z_0$, or $x_{i^*} = z_1$. This implies that either $\overline{H} \cup \{x\}$ and $\underline{H} \cup \{y\}$ are both connected subgraphs disjoint from one another, or $\underline{H} \cup \{y\}$ and $\overline{H} \cup \{x\}$ are both connected subgraphs disjoint from one another. Thus, the induction hypothesis applied to either $\underline{H} \cup \{x\}$ and $\overline{H} \cup \{y\}$ or $\underline{H} \cup \{y\}$ and $\overline{H} \cup \{x\}$ completes the proof.

Case $G \in [n]_e^3$: This proof is by induction. The result clearly holds for the case $\kappa = 1$. Assume the result holds for all $2l$-connected graphs, $2 \le l < \kappa$.

If $m_{i,j} \ge 3$ for some $(i,j,k) \in G$, then we can simply remove 2-connected graphs from $\{(i,j,s)\}_{s=1}^{\infty} \cap G$ until there are either 1 or 2 points in $\{(i,j,s)\}_{s=1}^{\infty} \cap G$. By this means, we can assume that $m_{i,j} \in \{1,2\}$ for all $(i,j,k) \in G$. If $m_{i,j} = 1$ for all $(i,j,k) \in G$, the result holds from the above work for graphs in $[n]_1^3$. Therefore, assume that $m_{i,j} = 2$ for some $(i,j,k) \in G$. Indeed, assume that $\mathbf{x} = \{(i,j,k_1),(i,j,k_2)\} \in G$. Then, by Corollary 4.1, $G \backslash \{\mathbf{x}\} = \overline{G}(\mathbf{x}) \cup \underline{G}(\mathbf{x})$, where $\overline{G}(\mathbf{x})$ and $\underline{G}(\mathbf{x})$ are connected subgraphs. If they touch, $\overline{G}(\mathbf{x}) \cup \underline{G}(\mathbf{x})$ is a connected graph whose magnitude is $2(\kappa - 1)$, and we may apply our induction assumption to complete the proof, noting that $\mathbf{x}$ is a 2-connected graph.

Suppose they are mutually separated. Then, their magnitudes are both even or odd. If they are both even, they have magnitudes strictly less than 2κ, and the proof is complete by using the induction on each of these sets. If they are both odd, then $\overline{G}(\mathbf{x}) \cup \{(i,j,k_1)\}$ and $\underline{G}(\mathbf{x}) \cup \{(i,j,k_2)\}$ both are connected and have even magnitudes strictly less that 2κ. Thus, we use our induction hypothesis to complete the proof. **Q.E.D.**

Corollary 4.2 *Let G be a $2k+1$-connected graph, $3 < 2k+1 \le |[n]_e^3|$. Then, there exists $k-1$ disjoint, 2-connected graphs, enumerated as $H_s^2(G)$, $s = 1, \ldots, k-1$ and a 3-connected graph $W^3(G)$, disjoint from $H_s^2(G)$, $s = 1, \ldots, k-1$, such that $W^3(G) \cup_{s=1}^{k-1} H_s^2(G) = G$.*

Proof. Referring to Lemma 4.7, let $x \in G$ such that x and $H(G)_1$ touch. Then, $H(G)$ has magnitude $2k$, and so by the above lemma there exists k mutually disjoint, 2-connected sets $W_1, \ldots, W_k$ such that $\cup_{i=1}^k W_i = H(G)$. Assume that $x \in W_k$. Then, the collection of sets $W_1, \cdots, W_{k-1}, W_k \cup H(G)_1$ fulfills the requirements of the corollary by letting $H_s^2(G) = W_s$ for $s = 1, \ldots, k-1$, and $W^3(G) = W_k \cup H(G)_1$. **Q.E.D.**

Corollary 4.3 *Let G be a $2k$-connected graph, $2 < k \le \frac{1}{2}|[n]_e^3|$. Then, there exists $k-2$ disjoint, 2-connected graphs, enumerated as $H_s^2(G)$, $s = 1, \ldots, k-2$ and a 4-connected graph $W^4(G)$, disjoint from $H_s^2(G)$, $s = 1, \ldots, k-2$, such that $W^4(G) \cup_{s=1}^{k-2} H_s^2(G) = G$*

Proof. By Lemma 4.8, there exists k mutually disjoint, 2-connected graphs, $H_i^2(G)$, $i = 1, \ldots, k$, such that $\cup_{i=1}^k H_i^2(G) = G$. Two of them must touch, as the graph is connected. **Q.E.D.**

Corollary 4.4 *Let n be so large that $4 < |[n]_e^3|$. Let G be a k-connected graph, $1 \le k \le 4$. Then, $G = \{x_1, \ldots, x_k\}$ such that*

$$x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_k \quad .$$

Proof. This is obvious for $k = 1,2,3$. When $k = 4$, we know by Lemma 4.8 that $G = \{x_1, x_2\} \cup \{y_1, y_2\}$ where $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. Since G is connected, $\{x_1, x_2\}$ and $\{y_1, y_2\}$ touch, say $x_2 \leftrightarrow y_1$ without loss of generality. Then,

$$x_1 \leftrightarrow x_2 \leftrightarrow y_1 \leftrightarrow y_2 \quad .$$

Q.E.D.

Before we proceed, note that $g_n^\kappa(\{\kappa\}) = g_n(\kappa)$ is just the set of all κ-connected graphs on $[n]_e^3$. Also, let

$$g_n^2(\{2\}; b-c) = \Big\{G \in \mathbf{A}_2^n : |G| = 2,\ G \text{ is connected and either balanced or cylindrical}\}.$$

Also, recall from Lemma 4.13, for $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \langle a_3, b_3\rangle \in \{0,1\}^2$,

$$\begin{aligned}
\lambda_n^1 &= \frac{1}{n^2}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2} \\
\lambda_n^2(a_1,b_1) &= \frac{1}{n^3}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)} \\
\lambda_n^3(a_1,b_2;a_2,b_2) &= \frac{1}{n^4}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\sum_{s_4=1}^{n} m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)} \\
\lambda_n^4(a_1,b_2;a_2,b_2;a_3,b_3) &= \frac{1}{n^5}\sum_{s_1,s_2=1}^{n} m_{s_1,s_2}\sum_{s_3=1}^{n} m_{h_{b_1}(g_{a_1}(s_1,s_2),s_3)}\sum_{s_4=1}^{n} m_{h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)} \times \\
&\quad \sum_{s_5=1}^{n} m_{h_{b_3}(g_{a_3}(h_{b_2}(g_{a_2}(h_{b_1}(g_{a_1}(s_1,s_2),s_3)),s_4)),s_5)} \quad .
\end{aligned}$$

Lemma 4.9 *There exists an N and a constant λ depending on N such that for all $n \geq N$,*

$$\begin{aligned}
|g_n^2(\{2\}; b-c)| &\leq \lambda n^2 && (4)\\
|g_n^2(\{2\})| &\leq \lambda n^3 && (5)\\
|g_n^3(\{3\})| &\leq \lambda n^4 && (6)\\
|g_n^4(\{4\})| &\leq \lambda n^5 && (7)
\end{aligned}$$

Proof.
Equation (4): By Corollary 4.4, any 2-connected graph may be constructed by choosing a point in $[n]_e^3$, and then choosing a point connected to it. There are $\sum_{\substack{i,j=1\\ i\neq j}}^{n} \sum_{k=1}^{m_{i,j}}$ possible ways to choose the first point. If the first point chosen is $(i,j,k) \in [n]_e^3$, to have the second

point be connected to it and the graph be balanced, the second point can be chosen in exactly $m_{j,i}$ ways. To have the second point be connected to it and the graph be cylindrical, the second point can be chosen in exactly $m_{i,j}-1$ ways. In either case, as $m_{i,j}=m_{j,i}$,

$$|g_n^2(\{2\};b-c)| \;\leq\; 2\sum_{\substack{i,j=1\\ i\neq j}}^{n} m_{i,j}^2 \leq 2(\sum_{i=1}^{n} m_i^2)^2 = n^2 2\beta^2(1+o_n^0(1)).$$

Let N_0 be so large that for all $n \geq N_0$, $|(1+o_n^0(1))| \leq 2$. Let $\lambda_0 = 2\beta^2$.

Equation (5): That

$$|g_n^2(\{2\})| \;=\; \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}((m_{i,j}-1)+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{i,s}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{s,j}+\sum_{\substack{s=1\\ s\neq i}}^{n} m_{s,i}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{j,s})$$

follows from simple counting. By Corollary 4.4, any 2-connected graph may be constructed by choosing a point in $[n]_e^3$, and then choosing a point connected to it. There are $\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}$ possible ways to choose the first point. If the first point chosen is $(i,j,k)\in[n]_e^3$, to have the second point be connected to it, the second point can be chosen in exactly

$$(m_{i,j}-1)+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{i,s}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{s,j}+\sum_{\substack{s=1\\ s\neq i}}^{n} m_{s,i}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{j,s}$$

ways. We divide by two, so as to discount for the ordering in this process.

Now, by Lemma 4.13, for some $\theta\in\mathbb{R}$, we have that

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}((m_{i,j}-1)+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{i,s}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{s,j}+\sum_{\substack{s=1\\ s\neq i}}^{n} m_{s,i}+\sum_{\substack{s=1\\ s\neq i,j}}^{n} m_{j,s})\\
&\qquad\leq\; \frac{1}{2}\sum_{i,j=1}^{n} m_{i,j}(\sum_{s=1}^{n} m_{i,s}+\sum_{s=1}^{n} m_{s,j}+\sum_{s=1}^{n} m_{s,i}+\sum_{s=1}^{n} m_{j,s})\\
&\qquad=\; \sum_{\langle a_1,b_1\rangle\in\{0,1\}^2} n^3\lambda^2(a_1,b_1)\\
&\qquad\leq\; n^3 4\theta(1+o_n^1(1))
\end{aligned}$$

Let N_1 be so large that for all $n\geq N_1$, $|(1+o_n^1(1))|\leq 2$. Let $\lambda_1 = 4\theta$.

Equation (6): As demonstrated above, for any point $(i,j,k)\in[n]_e^3$, a second point can be connected to it in exactly

$$m_{i,j}-1+(\sum_{\substack{s=1\\ s\neq j}}^{n} m_{i,s}+\sum_{\substack{s=1\\ s\neq i}}^{n} m_{s,j}+\sum_{\substack{s=1\\ s\neq j}}^{n} m_{s,i}+\sum_{\substack{s=1\\ s\neq i}}^{n} m_{j,s})$$

$$\leq \sum_{s=1}^{n} m_{i,s} + \sum_{s=1}^{n} m_{s,j} + \sum_{s=1}^{n} m_{s,i} + \sum_{s=1}^{n} m_{j,s}$$

From Corollary 4.4 it then follows, using the same reasoning as above, that all members of $g_n^3(\{3\})$ can be gotten by choosing points x_1 first, x_2 second, and x_3 third such that $x_1 \overset{t}{\leftrightarrow} x_2 \overset{t}{\leftrightarrow} x_3$ Therefore, using the same reasoning above, over counting by being able to choose some points more than once, and that noting that the above method will over count by at least 3, as it will produce the same graph when x_2 is first, x_3 is second, and x_1 is third or x_3 is first, x_1 is second, and x_2 is third, we have

$$\begin{aligned}
|g_n^3(\{3\})| \leq\ & \frac{1}{3}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{u=1}^{m_{i,s}}(m_{i,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{i,v} + \sum_{\substack{v=1\\ v\neq i}}^{n} m_{v,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{v,i} + \sum_{\substack{v=1\\ v\neq i}}^{n} m_{s,v})) + \\
& \frac{1}{3}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{u=1}^{m_{s,j}}(m_{s,j} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{j,v} + \sum_{\substack{v=1\\ v\neq j}}^{n} m_{v,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{v,j} + \sum_{\substack{v=1\\ v\neq j}}^{n} m_{s,v})) + \\
& \frac{1}{3}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{u=1}^{m_{s,i}}(m_{s,i} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{i,v} + \sum_{\substack{v=1\\ v\neq i}}^{n} m_{v,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{v,i} + \sum_{\substack{v=1\\ v\neq i}}^{n} m_{s,v})) + \\
& \frac{1}{3}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{u=1}^{m_{j,s}}(m_{j,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{j,v} + \sum_{\substack{v=1\\ v\neq j}}^{n} m_{v,s} + \sum_{\substack{v=1\\ v\neq s}}^{n} m_{v,j} + \sum_{\substack{v=1\\ v\neq j}}^{n} m_{s,v})) \\
\leq\ & \sum_{i,j=1}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{s=1}^{n}\sum_{u=1}^{m_{i,s}}(\sum_{v=1}^{n} m_{i,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,i} + \sum_{v=1}^{n} m_{s,v})) + \\
& \sum_{i,j=1}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{s=1}^{n}\sum_{u=1}^{m_{s,j}}(\sum_{v=1}^{n} m_{j,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,j} + \sum_{v=1}^{n} m_{s,v})) + \\
& \sum_{i,j=1}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{s=1}^{n}\sum_{u=1}^{m_{s,i}}(\sum_{v=1}^{n} m_{i,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,i} + \sum_{v=1}^{n} m_{s,v})) + \\
& \sum_{i,j=1}^{n}\sum_{k=1}^{m_{i,j}}(\sum_{s=1}^{n}\sum_{u=1}^{m_{j,s}}(\sum_{v=1}^{n} m_{j,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,j} + \sum_{v=1}^{n} m_{s,v})) \\
=\ & \sum_{i,j=1}^{n} m_{i,j}(\sum_{s=1}^{n} m_{i,s}(\sum_{v=1}^{n} m_{i,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,i} + \sum_{v=1}^{n} m_{s,v})) + \\
& \sum_{i,j=1}^{n} m_{i,j}(\sum_{s=1}^{n} m_{s,j}(\sum_{v=1}^{n} m_{j,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,j} + \sum_{v=1}^{n} m_{s,v})) +
\end{aligned}$$

$$\begin{aligned}
& \sum_{i,j=1}^{n} m_{i,j}(\sum_{s=1}^{n} m_{s,i}(\sum_{v=1}^{n} m_{i,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,i} + \sum_{v=1}^{n} m_{s,v})) + \\
& \sum_{i,j=1}^{n} m_{i,j}(\sum_{s=1}^{n} m_{j,s}(\sum_{v=1}^{n} m_{j,v} + \sum_{v=1}^{n} m_{v,s} + \sum_{v=1}^{n} m_{v,j} + \sum_{v=1}^{n} m_{s,v})) \\
= \; & n^4 \sum_{\langle a_1,b_1\rangle,\langle a_2,b_2\rangle \in \{0,1\}^2} \lambda^3(a_1,b_1;a_2,b_2) \\
\leq \; & n^4 16\theta(1+o_n^2(1))
\end{aligned}$$

again by Lemma 4.13.

For example, we can choose point 1 and 2, connected by being in the same row, and then choosing point 3 connected to point 2, which can happen in

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{k=1}^{m_{i,j}} (\sum_{\substack{s=1 \\ s\neq j}}^{n} \sum_{u=1}^{m_{i,s}} (\sum_{\substack{v=1 \\ v\neq s}}^{n} m_{i,v} + \sum_{\substack{v=1 \\ v\neq i}}^{n} m_{v,s} + \sum_{\substack{v=1 \\ v\neq s}}^{n} m_{v,i} + \sum_{\substack{v=1 \\ v\neq i}}^{n} m_{s,v}))$$

ways.

Let N_2 be so large that for all $n \geq N_2$, $|(1+o_n^2(1))| \leq 2$. Let $\lambda_2 = 16\theta$.

Equation (7): From Corollary 4.4 it then follows, using the fact that from Lemma 4.8 that a 4-connected graph may be decomposed into 2 2-connected graphs, and the same reasoning as above, that all members of $g_n^4(\{4\})$ can be gotten by choosing points x_1 first, x_2 second, x_3 third and x_4 fourth such that $x_1 \overset{t}{\leftrightarrow} x_2 \overset{t}{\leftrightarrow} x_3 \overset{t}{\leftrightarrow} x_4$. Therefore, using the same reasoning above, over counting by being able to choose some points more than once, and noting that the above method will over count by at least 4, Lemma 4.13 gives

$$\begin{aligned}
|g_n^4(\{4\})| \; & \leq \sum_{\langle a_1,b_1\rangle,\ldots,\langle a_3,b_3\rangle \in \{0,1\}^2} n^5 \lambda_n^4(a_1,b_1;a_2,b_2;a_3,b_3) \\
& \leq n^5 64\theta(1+o_n^3(1))
\end{aligned}$$

Let N_3 be so large that for all $n \geq N_3$, $|(1+o_n^3(1))| \leq 2$, and let $\lambda_3 = 64\theta$.

Then, setting $N = \max\{N_0, N_1, N_2, N_3\}$ and $\lambda = \max\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ completing the proof.

Q.E.D.

For $\kappa \in \mathbb{Z}^+$, define

$$\mathcal{P}^*_{[\kappa]} = \{\sigma \in \mathcal{P}_{[\kappa]} : r_s^\sigma \geq 2, s = 1, \ldots, \mathcal{C}_\sigma\}.$$

Lemma 4.10 *Let $n,\ \kappa \in \mathbb{Z}^+$, $\kappa \le |[n]_e^3|$. For $\sigma \in \mathcal{P}_{[\kappa]} \backslash \mathcal{P}^*_{[\kappa]}$,*

$$\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^{\kappa}(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})| \;=\; 0. \tag{8}$$

Also, for $\sigma \in \mathcal{P}^*_{[\kappa]}$, and $n,\ \kappa \in \mathbb{Z}^+$, $6 \le \kappa \le |[n]_e^3|$, $n \ge N$,

$$|g_n^{\kappa}(\sigma)| \;\le\; \frac{\lambda^{\kappa}}{\lfloor \kappa/6 \rfloor!} n^{(3\kappa)/2}, \tag{9}$$

where N and λ are defined in Lemma 4.9.

Proof.

Equation (8)

If $\sigma \in \mathcal{P}_{[\kappa]} \backslash \mathcal{P}^*_{[\kappa]}$, then $r_t^{\sigma} = 1$ for some $1 \le t \le \mathcal{C}_\sigma$. By Lemma 4.3, for all $G = \{(i_s, j_s, k_s)\}_{s=1}^{\kappa} \in g_n^{\kappa}(\sigma)$, $N(G) = \mathcal{C}_\sigma$, and

$$E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \;=\; \prod_{v=1}^{\mathcal{C}_\sigma} E(\prod_{(q,r,s)\in B(G)_v} Y_{q,r,s})$$

Then, by definition, for some $1 \le u \le \mathcal{C}_\sigma$, $|B(G)_u| = r_t^{\sigma} = 1$, implying that $B(G)_u = \{(\underline{i}, \underline{j}, \underline{k})\}$ for some $(\underline{i}, \underline{j}, \underline{k})$. Thus,

$$E(\prod_{(q,r,s)\in B(G)_u} Y_{q,r,s}) = E(Y_{\underline{i},\underline{j},\underline{k}}) = 0,$$

which implies equation (8).

Equation (9)

Let $n,\ \kappa \in \mathbb{Z}^+$, $6 \le \kappa \le |[n]_e^3|$, and $\sigma \in P^*_{[\kappa]}$. To simplify, let $\mathcal{C}_\sigma = l$, and $r_s^{\sigma} = r_s$ for $s = 1, \ldots, l$. For $s = 1, \ldots, l$, let m be the number of r_s that are odd. Note that $\kappa - 3m$ has to be even, and $3m \le \kappa$, which follows as such: the claim is obvious when $m = 0$. If $m \ge 1$ and $\{r_{s_i}\}_{i=1}^{m}$ is the collection of odd r_s, then $r_{s_i} - 3$ is even for all $1 \le i \le m$. But then, $\sum_{i=1}^{m}(r_{s_i} - 3)$ is even, and so,

$$\sum_{\substack{s=1 \\ s \notin \{s_i\}_{i=1}^{l}}}^{l} r_i + \sum_{i=1}^{m}(r_{s_i} - 3) = \kappa - 3m$$

is even.

Now, consider the following simple algorithm, hereafter referred to algorithm R. In this algorithm, if $m = 0$, it is understood that steps 3 and 4 are not performed, and if $3m = \kappa$, that steps 1 and 2 are not performed:

1. Step 1: Choose a 2-connected graph.

Let $2 \le i \le \frac{\kappa-3m}{2}$.

2. Step i: Assuming the 2-connected graphs $G_1^2, \dots, G_{i-1}^2$ have been chosen mutually disjoint, choose a 2-connected graph from $[n]_e^3 \setminus \cup_{j=1}^{i-1} G_j^2$.

3. Step $\frac{\kappa-3m}{2}+1$: Choose a 3-connected graph G_1^3 from $[n]_e^3 \setminus \cup_{j=1}^{\frac{\kappa-3m}{2}} G_j^2$.

Let $\frac{\kappa-3m}{2}+1 \le q \le \frac{\kappa-3m}{2}+m$.

4. Step q: Let $s = q - \frac{\kappa-3m}{2}$. Assuming the 2-connected graphs $G_1^2, \dots, G_{\frac{\kappa-3m}{2}}^2$, and the 3-connected graphs $G_1^3, \dots, G_{s-1}^3$ have been chosen mutually disjoint, choose a 3-connected graph G_s^3 from $[n]_e^3 \setminus \cup_{i=1}^{\frac{\kappa-3m}{2}} G_i^2 \cup \bigcup_{j=1}^{s-1} G_j^3$.

5. Step $\frac{\kappa-3m}{2}+m+1$: Collect the resulting $\frac{\kappa-3m}{2}+m$ connected graphs.

Note that $\sum_{i=1}^{\frac{\kappa-3m}{2}} |G_i^2| + \sum_{j=1}^{m} |G_j^3| = 2\frac{\kappa-3m}{2} + 3m = \kappa \le |[n]_e^3|$, which implies the algorithm is possible.

First, assume that $0 < 3m < \kappa$. By Lemma 4.8 and Corollary 4.2, it follows that that any element of $\{\{B(G)_1, \dots, B(G)_{N(G)}\}\}_{G \in g_n^\kappa(\sigma)}$ can be achieved with this algorithm. But how many unordered different collections of connected graphs will result from the above process?

From simple counting, an upper bound on this number is

$$\Big(\frac{1}{((\kappa-3m)/2)!}(\lambda n^3)^{(\kappa-3m)/2}\frac{1}{m!}(\lambda n^4)^m\Big) \quad \le \quad \frac{\lambda^\kappa}{(\kappa-3m)/2)!m!}n^{(1/2)(3\kappa-m)}.$$

From Lemma 4.9, and as already discussed, there are at most λn^3 2-connected graphs. We are picking $(\kappa-3m)/2$ of them, but the order of choosing is inconsequential, which we divide by $((\kappa-3m)/2)!$. From Lemma 4.9, there are at most λn^4 3-connected graphs. We are picking m 3-connected graphs, but, again, the order of choosing is inconsequential, so we divide by $m!$.

Since $(\kappa-3m)/2 + m = (\kappa-m)/2$, either $(\kappa-3m)/2$ or m is greater than or equal to $(\kappa-m)/4$. Thus,

$$((\kappa-3m)/2)!m! \ge \lfloor(\kappa-m)/4\rfloor! \ge \lfloor(\kappa-(1/3)\kappa)/4\rfloor! = \lfloor\kappa/6\rfloor!.$$

Thus, the result holds for $0 < 3m < \kappa$.

Now assume that $m = 0$. In this case, an upperbound for the number of different collections of connected graphs that will result from the above process is bounded above with the same reasoning as used above, except $m = 0$. Since $6 \le \kappa$, this gives that

$$|g_n^\kappa(\sigma)| \le \frac{(\lambda n^3)^{\kappa/2}}{(\kappa/2)!} \le \frac{\lambda^\kappa}{\lfloor\kappa/6\rfloor!}n^{(3\kappa)/2}.$$

Finally, assume that $3m = \kappa$. In this case, $l = m$, $r_i = 3$ for all i. From Lemma 4.9, and as seen above, there are at most λn^4 ways to pick a 3-connected graph on $[n]_e^3$. This gives that, for all $6 \le \kappa$,

$$|g_n^\kappa(\sigma)| \le \frac{\lambda^{\kappa/3}}{(\kappa/3)!} n^{(4\kappa)/3} \le \frac{\lambda^\kappa}{\lfloor \kappa/6 \rfloor!} n^{(3\kappa)/2}.$$

Q.E.D.

4.4.2 Proof of Proposition

Proposition 2.5 *Let $n,\ \kappa \in \mathbb{Z}^+$, $6 \le \kappa \le |[n]_e^3|$. There exists a function $f : \mathbb{Z}^+ \to \mathbb{R}^+$ such that*

$$\sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})| \frac{|t|^\kappa}{n^{(3\kappa)/2}} \le f(\kappa),$$

and

$$\sum_{\kappa=6}^\infty f(\kappa) < \infty.$$

Proof. Let $n,\ k \in \mathbb{Z}^+$, $6 \le k \le |[n]_e^3|$. From equation (8) of Lemma 4.10, it follows that

$$\sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})| = \sum_{\sigma \in \mathcal{P}^*_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})|.$$

From Lemma 4.10, and that $|\mathcal{P}^*_{[\kappa]}| \le |\mathcal{P}_{[\kappa]}| = p(\kappa) \le 2^\kappa$, it follows that

$$\begin{aligned}
\sum_{\sigma \in \mathcal{P}^*_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} |E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})| \frac{|t|^\kappa}{n^{(3\kappa)/2}} &\le \sum_{\sigma \in \mathcal{P}^*_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} \frac{C^\kappa |t|^\kappa}{n^{(3\kappa)/2}} \\
&= \frac{C^\kappa |t|^\kappa}{n^{(3\kappa)/2}} \sum_{\sigma \in \mathcal{P}^*_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} \\
&= \frac{C^\kappa |t|^\kappa}{n^{(3\kappa)/2}} \sum_{\sigma \in \mathcal{P}^*_{[\kappa]}} |g_n^\kappa(\sigma)| \\
&\le \frac{C^\kappa |t|^\kappa}{n^{(3\kappa)/2}} 2^\kappa \frac{\lambda^\kappa}{\lfloor (\kappa/6) \rfloor!} n^{(3\kappa)/2} \\
&\le \frac{(2\lambda)^\kappa}{\lfloor \kappa/6 \rfloor!} C^\kappa |t|^\kappa.
\end{aligned}$$

If we let $f(\kappa) = \frac{(2\lambda)^\kappa}{\lfloor \kappa/6 \rfloor !} C^\kappa |t|^\kappa$ for $6 \leq \kappa$, then it is not hard to show that

$$\sum_{\kappa=6}^{\infty} f(\kappa) < \infty.$$

By Stirling's approximation, let $M > 2$ be so large such that, for all $\kappa \geq M$,

$$\begin{aligned} \lfloor (\kappa/6) \rfloor ! &\geq \lceil (\kappa/12) \rceil ! \\ &\geq \frac{1}{2} (\lceil (\kappa/12) \rceil / e)^{\lceil (\kappa/12) \rceil} \sqrt{2\pi \lceil (\kappa/12) \rceil} \\ &\geq \frac{1}{2} ((\kappa/12)/e)^{(\kappa/12)}. \end{aligned}$$

Then, with $\beta = 2\lambda C|t|$,

$$\begin{aligned} \frac{1}{2} \sum_{\kappa=M}^{\infty} \frac{\beta^\kappa}{\lfloor (\kappa/6) \rfloor !} &\leq \frac{1}{2} 2 \sum_{\kappa=M}^{\infty} \Big(\frac{(12e)^{1/12} \beta}{\kappa^{1/12}} \Big)^\kappa \\ &\leq \int_{M-1}^{\infty} \Big(\frac{(12e)^{1/12} \beta}{x^{1/12}} \Big)^x dx < \infty. \end{aligned}$$

Q.E.D.

4.5 Proof of Proposition 2.6

We require 3 preliminary lemmas. We review relevant notation before each lemma.

Recall that for any finite graph $G \in [n]_e^3$, we defined

$$V_t(G) = \{(i,j,k) \in [n]_e^3 : (i,j,k) \leftrightarrow (q,r,s) \text{ for some } (q,r,s) \in G\}.$$

Lemma 4.11 *Let $n \geq 3$. If G is an unbalanced, non-cylindrical, 2-connected graph on $[n]_e^3$, then there exist $A \subset \{1, \ldots, n\}$ such that $|A| = 3$, and $V_t(G) = A \times \mathbb{Z}^+ \times \mathbb{Z}^+ \cup \mathbb{Z}^+ \times A \times \mathbb{Z}^+ \cap Z_{m,+}^3$. If G is either a balanced, 2-connected graph on $[n]_e^3$ or a 2-connected cylinder on $[n]_e^3$, then there exist $A \subset \{1, \ldots, n\}$ such that $|A| = 2$, and $V_t(G) = A \times \mathbb{Z}^+ \times \mathbb{Z}^+ \cup \mathbb{Z}^+ \times A \times \mathbb{Z}^+ \cap Z_{m,+}^3$.*

In other words, when G is an unbalanced, non-cylindrical 2-connected set, then $V_t(G)$ are all points that are either in one of three different hyper-rows or one of three different hyper-columns.

Solution. This is almost obvious. Let G be an unbalanced, non-cylindrical, 2-connected graph on $[n]_e^3$. Then, we may assume $G = \{(i,j,k), (q,r,s)\}$ where $\{i,j,q,r\}$ contains exactly

3 unique elements, denote them i_1, i_2, i_3. It follows that if $A = \{i_1, i_2, i_3\}$, then $V_t(G) = A \times \mathbb{Z}^+ \times \mathbb{Z}^+ \cup \mathbb{Z}^+ \times A \times \mathbb{Z}^+ \cap Z^3_{m,+}$. A similar argument holds when G is either a balanced, 2-connected graph on $[n]^3_e$ or a a 2-connected cylinder on $[n]^3_e$. **Q.E.D.**

Let $n \in \mathbb{Z}^+$, n large. Let $\mathbf{X} = \{x_{(i,j,k),(q,r,s)}\}_{i,j,k,\ q,r,s \in [n]^3_e}$ be a finite set of distinct objects, e.g. $x_{(i,j,k),(q,r,s)}$ may be a 2-graph defined by the indices of a pair of variables $Y_{(i,j,k)}$, $Y_{(q,r,s)}$, i.e. $x_{(i,j,k),(q,r,s)} = \{(i,j,k),(q,r,s)\}$. For $i,\ j \in \mathbb{Z}^+,\ i < j$, and $\mathbf{x}_j = \langle x_1, \ldots, x_j\rangle \in (\mathbf{X})^j$, we define

$$\mathbf{x}_{j,i} \quad = \quad \langle x_1, \ldots, x_i\rangle.$$

Let $T = \{t_1, t_2, t_3, t_4\}$ be a collection of 4 distinct real numbers. Let $f : \mathbf{X} \to T = \{t_1, t_2, t_3, t_4\} \subset \mathbb{R}$, so the t_i represents one of 4 distinct categories that an element of $\mathbf{X}$ may belong to, e.g. colors, or, in our case, possible covariance values a correlated pair $\{Y_{i,j,k}, Y_{q,r,s}\}$ may take.

Lemma 4.12 *Let $m \in \mathbb{Z}^+,\ m \geq 2$, and $\mathcal{C} \subset (\mathbf{X})^m$. For $1 \leq i \leq m$, let*

$$\mathcal{C}_i \quad = \quad \{\mathbf{z}_i \in (\mathbf{X})^i : \mathbf{x}_{m,i} = \mathbf{z}_i \text{ for some } \mathbf{x}_m \in \mathcal{C}\}$$

For $t \in T$, let

$$A_1(t) \quad = \quad \{x \in \mathcal{C}_1 :\ f(x_1) = t\},$$

and for $2 \leq i \leq m$ and $\mathbf{z}_{i-1} \in \mathcal{C}_{i-1}$, let

$$A_i(\mathbf{z}_{i-1}; t) = \quad \{\mathbf{x}_i \in \mathcal{C}_i : \mathbf{x}_{i,i-1} = \mathbf{z}_{i-1},\ f(x_i) = t\}$$

Assume that there exists a $\mathbb{Z}^+$-valued sequence $\{\epsilon_n\}_{n\geq 1}$ such that $\epsilon_n = o_n(n^3)$, and for all $z_1 \in \mathcal{C}_1$,

$$||A_1(t_j)| - \alpha^2\beta\frac{n^3}{2}| \quad \leq \quad \epsilon_n$$

for $1 \leq j \leq 4$. Further, assume that for $2 \leq i \leq m$ and any $\mathbf{x}_i \in \mathcal{C}_i$,

$$||A_i(\mathbf{x}_{i-1}; t_j)| - \alpha^2\beta\frac{n^3}{2}| \quad \leq \quad \epsilon_n$$

for $1 \leq j \leq 4$.

Then, for any $m \in \mathbb{Z}^+$, we have that

$$\sum_{\langle z_1,\ldots,z_m\rangle \in \mathcal{C}} f(z_1)\cdots f(z_m) \quad = \quad (\frac{\alpha^2\beta(t_1+t_2+t_3+t_4)}{2})^m n^{3m} + o_n(n^{3m}).$$

Solution. Let $\hat{\mathbf{t}}_m = \langle \hat{t}_1, \ldots, \hat{t}_m \rangle \in T^m$, and

$$\begin{aligned} \mathcal{C}_m(\hat{\mathbf{t}}_m) &= \{\mathbf{x}_m \in \mathcal{C} : \; f(x_i) = \hat{t}_i, \; i = 1, \ldots, m\} \\ \Delta(\hat{\mathbf{t}}_m) &= \{\langle x_1^1, \mathbf{x}_2^2, \ldots, \mathbf{x}_m^m \rangle \in \prod_{i=1}^{m} \mathcal{C}_i : \\ & \qquad x_1^1 \in A_1(\hat{t}_1), \; \mathbf{x}_2^2 \in A_2(x_1^1; \hat{t}_2), \ldots, \mathbf{x}_m^m \in A_m(\mathbf{x}_{m-1}^{m-1}; \hat{t}_m)\} \end{aligned}$$

There is a bijection between $\mathcal{C}_m(\hat{\mathbf{t}}_m)$ and $\Delta(\hat{\mathbf{t}}_m)$. To see such consider, the following map $g : \Delta(\hat{\mathbf{t}}_m) \to (\mathbf{X})^m$ such that for $\langle x_1^1, \mathbf{x}_2^2, \ldots, \mathbf{x}_m^m \rangle \in \Delta(\hat{\mathbf{t}}_m)$

$$g(\langle x_1^1, \mathbf{x}_2^2, \ldots, \mathbf{x}_m^m \rangle) = \langle x_1^1, x_2^2, \ldots, x_m^m \rangle$$

By construction, g is into $\mathcal{C}_m(\hat{\mathbf{t}}_m)$, and it must be onto. For $\mathbf{x}_m \in \mathcal{C}_m(\hat{\mathbf{t}}_m)$, it follows that

$$\langle x_1, \mathbf{x}_{m,2}, \ldots, \mathbf{x}_{m,m} \rangle \in \Delta(\hat{\mathbf{t}}_m), \; g(\langle x_1, \mathbf{x}_{m,2}, \ldots, \mathbf{x}_{m,m} \rangle) = \mathbf{x}_m.$$

It also clearly has to be 1-1. Assume $\langle x_1^1, \mathbf{x}_2^2, \ldots, \mathbf{x}_m^m \rangle \neq \langle \hat{x}_1^1, \hat{\mathbf{x}}_2^2, \ldots, \hat{\mathbf{x}}_m^m \rangle$ and they are both members of $\Delta(\hat{\mathbf{t}}_m)$. If i is the smallest index such that $\mathbf{x}_i^i \neq \hat{\mathbf{x}}_i^i$, then $x_i^i \neq \hat{x}_i^i$, and their images under g must be distinct.

We now may find an upper and lower bound on $|\mathcal{C}_m(\hat{\mathbf{t}}_m)|$. As $\mathcal{C}_m(\hat{\mathbf{t}}_m)$ and $\Delta(\hat{\mathbf{t}}_m)$ are in $1-1$ correspondence, it follows that

$$|\mathcal{C}_m(\hat{\mathbf{t}}_m)| = \sum_{x_1 \in A_1(\hat{t}_1)} \sum_{\mathbf{x}_2^2 \in A_2(x_1; \hat{t}_2)} \cdots \sum_{\mathbf{x}_{m-1}^{m-1} \in A_{m-1}(\mathbf{x}_{m-2}^{m-2}; \hat{t}_m)} \sum_{\mathbf{x}_m^m \in A_m(\mathbf{x}_{m-1}^{m-1}; \hat{t}_m)}$$

By assumption, since

$$(\alpha^2 \beta \frac{n^3}{2} - \epsilon_n) \leq \sum_{\mathbf{x}_i \in A_i(x_{i-1}; \hat{t}_i)} \leq (\alpha^2 \beta \frac{n^3}{2} + \epsilon_n).$$

for all $\mathbf{x}_{i-1} \in \mathcal{C}_i$, we have that

$$(\alpha^2 \beta \frac{n^3}{2} - \epsilon_n)^m \leq \sum_{x_1 \in A_1(\hat{t}_1)} \sum_{\mathbf{x}_2 \in A_2(x_1; \hat{t}_2)} \cdots \sum_{\mathbf{x}_m \in A_m(\mathbf{x}_{m-1}; \hat{t}_m)} \leq (\alpha^2 \beta \frac{n^3}{2} + \epsilon_n)^m.$$

As

$$(\alpha^2 \beta \frac{n^3}{2} \pm \epsilon_n)^m = \sum_{i=0}^{m} \binom{m}{i} (\alpha^2 \beta \frac{n^3}{2})^{m-k} (\pm \epsilon_n)^k = (\alpha^2 \beta \frac{n^3}{2})^m + o_n(n^{3m}),$$

it follows that $|\mathcal{C}_m(\hat{\mathbf{t}}_m)| = (\alpha^2 \beta \frac{n^3}{2})^m + o_n(n^{3m})$.

From this we may conclude that

$$\sum_{\mathbf{z}_m \in \mathcal{C}_m(\hat{\mathbf{t}}_m)} f(x_1)\cdots f(x_m) \;=\; (\hat{t}_1\cdots\hat{t}_m)(\alpha^2\beta\frac{n^3}{2})^m + o_n(n^{3m}).$$

Finally, we have

$$\begin{aligned}
\sum_{\langle z_1,\ldots,z_m\rangle\in\mathcal{C}} f(z_1)\cdots f(z_m) &= \sum_{\hat{\mathbf{t}}_m\in T^m}\ \sum_{\langle z_1,\ldots,z_m\rangle\in\mathcal{C}_m(\hat{\mathbf{t}}_m)} f(z_1)\cdots f(z_m)\\
&= \sum_{\hat{\mathbf{t}}_m\in T^m}\left\{(\hat{t}_1\cdots\hat{t}_m)(\alpha^2\beta\frac{n^3}{2})^m + o_n(n^{3m})\right\}\\
&= (\sum_{\hat{\mathbf{t}}_m\in T^m}(\hat{t}_1\cdots\hat{t}_m))(\alpha^2\beta\frac{n^3}{2})^m + o_n(n^{3m})\\
&= (t_1+t_2+t_3+t_4)^m(\alpha^2\beta\frac{n^3}{2})^m + o_n(n^{3m}).
\end{aligned}$$

Q.E.D.

Lemma 4.13 *Let* $k\in\mathbb{Z}^+$, *and* $\{a_i\}_{i=1}^{\infty}\in(\mathbb{Z}^+)^{\mathbb{Z}^+}$ *such that* $\frac{1}{n}\sum_{i=1}^n a_i \to a$.

$$\frac{\max_{A_n\subset[n],|A_n|=k}\sum_{i\in A_n} a_i}{n} \;\to\; 0 \;\;.$$

Proof. Let $i_n\in[n]$ such that $\max_{i=1}^n a_i = a_{i_n}$. If $\lim_n i_n<\infty$, then the result is obvious. Suppose that $i_n\to\infty$. Then, we have that

$$a = \lim_n \frac{1}{i_n}\sum_{i=1}^{i_n} a_i = \lim_n \frac{i_n-1}{i_n}\frac{1}{i_n-1}\sum_{i=1}^{i_n-1} a_i + \lim_n \frac{a_{i_n}}{i_n} = a + \lim_n \frac{a_{i_n}}{i_n},$$

which implies that

$$0 = \lim_n \frac{a_{i_n}}{i_n} \ge \lim_n \frac{a_{i_n}}{n} \ge 0 \;\;.$$

Thus,

$$\frac{\max_{A_n\subset[n],|A_n|=k}\sum_{i\in A_n} a_i}{n} \le \frac{k a_{i_n}}{n} \to 0 \;\;.$$

Q.E.D.

Corollary 4.5 *Let $k \in \mathbb{Z}^+$, and $\{a_i\}_{i=1}^{\infty} \in (\mathbb{Z}^+)^{\mathbb{Z}^+}$ such that $\frac{1}{n}\sum_{i=1}^{n} a_i \to a$.*

$$\max_{A_n \subset [n], |A_n|=k} |\frac{1}{n}\sum_{i=1}^{n} a_i - \frac{1}{n}\sum_{\substack{i=1\\ i\notin A_n}}^{n} a_i| \quad \to \quad 0 \quad .$$

Solution.

$$\max_{A_n \subset [n], |A_n|=k} |\frac{1}{n}\sum_{i=1}^{n} a_i - \frac{1}{n}\sum_{\substack{i=1\\ i\notin A_n}}^{n} a_i| = \frac{\max_{A_n \subset [n], |A_n|=k} \sum_{i\in A_n} a_i}{n} \to 0 \quad .$$

Q.E.D.

Let $A,\ B \subset \mathbb{Z}^+$. Recall that we call an $(|A|\times|B|)$-array complete or a complete $(|A|\times|B|)$-array iff $m_{i,j} > 0$ for $i \in A,\ j \in B$.

Lemma 4.14 *Let $n,\ \kappa \in \mathbb{Z}^+$, $\kappa \leq |[n]_e^3|$. Then, for $\sigma \in P^*_{[\kappa]}$, n sufficiently larger than κ,*

$$\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa} \in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})$$
$$= \begin{cases} \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} n^{(3\kappa)/2}(1+o_n(1)) & : \quad r_s^{\sigma} = 2,\ s = 1,\dots,\mathcal{C}_{\sigma}, \quad (*)\\ o_n(n^{(3\kappa)/2}) & : \quad r_s^{\sigma} \neq 2 \textit{ for some } 1 \leq s \leq \mathcal{C}_{\sigma}. \quad (**) \end{cases}$$

Proof. Let $\mathcal{C}_{\sigma} = l$, and $r_s^{\sigma} = r_s$ for $s = 1,\dots,l$.

Equation $(*)$

Let $n \geq N$ from Lemma 4.9. Let $r_i = 2$ for all $i = 1,\dots,l$. Then, κ is even, $l = \kappa/2$, and $N(G) = \frac{\kappa}{2}$ for all $G \in g_n^{\kappa}(\sigma)$. Let $\tilde{n} = |\{m_i, 1 \leq i \leq n : m_i \neq 0\}|$. Assume that n is so large that $3\frac{\kappa}{2} \leq \tilde{n}$. This is possible because $\tilde{n} \xrightarrow{n} \infty$, as $\lim_n \frac{1}{n}\sum_{i=1}^{n} m_i \to \alpha > 0$. Consider the following simple algorithm, henceforth referred to as algorithm Q:

1. Step 1: Choose a 2-connected graph G_1^2.

Let $2 \leq k \leq \kappa/2$.

2. Step k: Assuming the 2-connected graphs $G_1^2,\dots,G_{k-1}^2$ have been chosen mutually separated, eliminate $V_t(G_j^2)$, $1 \leq j \leq k-1$. Choose a 2-connected graph G_k^2 from the remaining points on the $[n]_e^3$.
3. Step $\kappa/2+1$: Form the ordered collection of $\kappa/2$ mutually separated, 2-connected graphs $\langle G_1^2,\dots,G_{\kappa/2}^2\rangle$.
4. Step $\kappa/2+2$: Form the unordered collection of $\kappa/2$ mutually separated, 2-connected graphs $\{G_j^2\}_{j=1}^{\kappa/2}$.

Algorithm Q is possible because for $1 \leq k < \kappa/2$, during the $k+1$th step, by Lemma 4.11, at most $3k$ hyper-rows and $3k$ hyper-columns are removed, leaving at least a complete $(\tilde{n}-3k)\times(\tilde{n}-3k)$-array of points G to be chosen from. However, since $\tilde{n}-3k \geq \tilde{n}-3(\kappa/2-1)) \geq 3$, and clearly a 2-connected graph can be chosen from a complete $s \times s$-array for any $s \geq 3$, a 2-connected graph may be chosen from G.

Let Ξ_Q^o be the set of all ordered collections of graphs that may be generated by algorithm Q. Let Ξ_Q^u be the set of all unordered collections of graphs that may be generated by algorithm Q. Let S_Q^o be all elements of Ξ_Q^o that have either a balanced marginal graph or a cylindrical marginal graph, and S_Q^u be all elements of Ξ_Q^u that have either a balanced marginal graph or a cylindrical marginal graph. Specifically,

$$\begin{aligned}
\Xi_Q^o &= \{\langle G_1,\ldots,G_{\kappa/2}\rangle : \langle G_1,\ldots,G_{\kappa/2}\rangle \text{ may be generated by algorithm } Q\},\\
\Xi_Q^u &= \{\{G_1,\ldots,G_{\kappa/2}\} : \{G_1,\ldots,G_{\kappa/2}\} \text{ may be generated by algorithm } Q\},\\
S_Q^o &= \{\langle G_1,\ldots,G_{\kappa/2}\rangle \in \Xi_Q^o : \text{ for some } 1 \leq i \leq \kappa/2,\ G_i \text{ is either balanced or cylindrical}\},\\
S_Q^u &= \{\{G_1,\ldots,G_{\kappa/2}\} \in \Xi_Q^u : \text{ for some } 1 \leq i \leq \kappa/2,\ G_i \text{ is either balanced or cylindrical}\}.
\end{aligned}$$

It is useful to remark that $|\Xi_Q^o|/(\kappa/2)! = |\Xi_Q^u|$, $|S_Q^o|/(\kappa/2)! = |S_Q^u|$.

Now, clearly

$$\Xi_Q^u \subset \{\{B(G)_1,\ldots,B(G)_{N(G)}\}\}_{G\in g_n^\kappa(\sigma)},$$

and if $\{B(G)_1,\ldots,B(G)_{\kappa/2}\} \in \{\{B(G)_1,\ldots,B(G)_{N(G)}\}\}_{G\in g_n^\kappa(\sigma)}$, then it follows that

$$V_t(B(G)_i) \cap V_t(B(G)_j) = \emptyset$$

for all $1 \leq i \neq j \leq \kappa/2$. Thus, any element of $\{\{B(G)_1,\ldots,B(G)_{N(G)}\}\}_{G\in g_n^\kappa(\sigma)}$ may be generated by algorithm Q, and so,

$$\Xi_Q^u = \{\{B(G)_1,\ldots,B(G)_{N(G)}\}\}_{G\in g_n^\kappa(\sigma)}.$$

Now, if for any 2-connected set $G = \{(i,j,k),(q,r,s)\} \in [n]_e^3$, we will let $\mathbf{Y}_G = Y_{i,j,k}\cdot Y_{q,r,s}$, it then follows from our previous discussion that

$$\begin{aligned}
\sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa}) &= \sum_{\{G_1,\ldots,G_{\kappa/2}\}\in\{\{B(G)_1,\ldots,B(G)_{N(G)}\}\}_{G\in g_n^\kappa(\sigma)}} E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}})\\
&= \sum_{\{G_1,\ldots,G_{\kappa/2}\}\in\Xi_Q^u} E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}})
\end{aligned}$$

We now want to control the sum of the expected values over Ξ_Q^u, which we accomplish as follows: recall that we are assuming that n is so large that $3\kappa \leq \tilde{n}$, and consider the following simple algorithm, which will be referred to as algorithm L:

1. Step 1: Choose an unbalanced, non-cylindrical, 2-connected graph H_1^2.

Let $2 \leq k \leq \kappa/2$.

2. Step k: Assuming the unbalanced, non-cylindrical, 2-connected graphs $H_1^2, \ldots, H_{k-1}^2$ have been chosen mutually separated, eliminate eliminate $V_t(H_j^2)$, $1 \leq j \leq k-1$. Choose an unbalanced, non-cylindrical, 2-connected graph H_i^2 from the remaining points on the $[n]_e^3$.

3. Step $\kappa/2+1$: Form the ordered collection of $\kappa/2$ mutually separated, 2-connected graphs $\langle H_1^2, \ldots, H_{\kappa/2}^2 \rangle$.

4. Step $\kappa/2+2$: Form the unordered collection of $\kappa/2$ mutually separated, 2-connected graphs $\{H_j^2\}_{j=1}^{\kappa/2}$.

Algorithm L for exactly the same reason that algorithm Q is possible.

Let Ξ_L^o be the set of all ordered collections of sets that may be generated by algorithm L. Let Ξ_L^u be the set of all unordered collections of sets that may be generated by algorithm L. Specifically,

$$\begin{aligned}
\Xi_L^o &= \{\langle H_1, \ldots, H_{\kappa/2} \rangle : \langle H_1, \ldots, H_{\kappa/2} \rangle \text{ may be generated by algorithm } L\}, \\
\Xi_L^u &= \{\{H_1, \ldots, H_{\kappa/2}\} : \{H_1, \ldots, H_{\kappa/2}\} \text{ may be generated by algorithm } L\}.
\end{aligned}$$

It is useful to remark that $|\Xi_L^o|/(\kappa/2)! = |\Xi_L^u|$.

From construction, we have that

$$\Xi_L^u \subset \Xi_Q^u = \{\{B(G)_1, \ldots, B(G)_{N(G)}\}\}_{G \in g_n^\kappa(\sigma)}.$$

Furthermore, we have that

$$|\Xi_Q^u| - |\Xi_L^u| = |S_Q^u| \leq |S_Q^o|$$

However, it follows by choosing first at which step in algorithm Q will either a balanced, 2-connected graph be chosen or a cylindrical, 2-connected graph be chosen, and then using Lemma 4.9 to count the number of possible 2-connected graphs at each step, that

$$|S_Q^o| \;\leq\; \sum_{i=1}^{\kappa/2} \binom{\kappa/2}{i} (\lambda n^2)^i (\lambda n^3)^{\kappa/2-i} = o_n(n^{(3\kappa)/2}).$$

From this, it immediately follows that

$$\begin{aligned}
\sum_{\{G_1, \ldots, G_{\kappa/2}\} \in \Xi_Q^u} E(\mathbf{Y}_{G_1} \cdots \mathbf{Y}_{G_{\kappa/2}}) &= \sum_{\{G_1, \ldots, G_{\kappa/2}\} \in \Xi_L^u} E(\mathbf{Y}_{G_1} \cdots \mathbf{Y}_{G_{\kappa/2}}) + |C|^\kappa o_n(n^{(3\kappa)/2}) \\
&= \sum_{\{G_1, \ldots, G_{\kappa/2}\} \in \Xi_L^u} E(\mathbf{Y}_{G_1} \cdots \mathbf{Y}_{G_{\kappa/2}}) + o_n(n^{(3\kappa)/2})
\end{aligned}$$

Therefore, it is left to show that

$$\sum_{\{G_1,\ldots,G_{\kappa/2}\}\in\Xi^u_L} E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}}) = \frac{1}{(\kappa/2)!}(\frac{\rho+\tau+\nu+\eta}{2})^{\kappa/2} n^{(3\kappa)/2}(1+o_n(1)).$$

We start this by noting that every member $\{G_1,\ldots,G_{\kappa/2}\}$ of Ξ^u_L corresponds to exactly $(\kappa/2)!$ members of Ξ^o_L, each of which is just an ordered permutation of $\{G_1,\ldots,G_{\kappa/2}\}$. Then, if σ is a permutation of $[\kappa/2]$, because

$$E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}}) = E(\mathbf{Y}_{G_{\sigma(1)}}\cdots\mathbf{Y}_{G_{\sigma(\kappa/2)}})$$

it follows that

$$\begin{aligned}\sum_{\{G_1,\ldots,G_{\kappa/2}\}\in\Xi^u_L} E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}}) &= \frac{1}{(\kappa/2)!}\sum_{\langle G_1,\ldots,G_{\kappa/2}\rangle\in\Xi^o_L} E(\mathbf{Y}_{G_1}\cdots\mathbf{Y}_{G_{\kappa/2}})\\ &= \frac{1}{(\kappa/2)!}\sum_{\langle G_1,\ldots,G_{\kappa/2}\rangle\in\Xi^o_L} E(\mathbf{Y}_{G_1})\cdots E(\mathbf{Y}_{G_{\kappa/2}})\end{aligned}$$

We now apply Lemma 4.12. Recall for $t\in\{\rho,\tau,\nu,\eta\}$, that an unbalanced, non-cylindrical, 2-connected graph G is of type t or a type t 2-connected graph iff $E(\mathbf{Y}_G)=t$. Let $\mathcal{C}=\Xi^o_L$. Then, $\mathcal{C}_k$ is the collection of all possible ordered collections of unbalanced, non-cylindrical, 2-connected graphs $\langle H^2_1,\ldots,H^2_k\rangle$ that may be generated by algorithm L up to step k, $1\le k\le\kappa/2$. In addition, the f of Lemma 4.12 in this case is the map that takes any of the connected 2-graphs H^2_k to $f(H^2_k)=E(\mathbf{Y}_{H^2_k})$, so that $T=\{t_1=\rho,t_2=\tau,t_3=\nu,t_4=\eta\}$. This is because the H^2_k in this case are unbalanced, non-cylindrical, 2-connected graphs.

Now, for $1\le j\le 4$, $A_1(t_j)$, as defined in Lemma 4.12 in reference to $\mathcal{C}=\Xi^o_L$, is simply the number of H^2_1 generated from algorithm L that are of type t_j, and for $2\le k\le\kappa/2$, and any $\mathbf{x}_{k-1}=\langle\underline{H}^2_1,\ldots,\underline{H}^2_{k-1}\rangle\in\mathcal{C}_{k-1}$, $A_k(\mathbf{x}_{k-1};t_j)$ as defined in in Lemma 4.12 in reference to $\mathcal{C}=\Xi^o_L$, is simply the number of $\langle H^2_1,\ldots,H^2_k\rangle$ generated from algorithm L such that $\langle H^2_1,\ldots,H^2_{k-1}\rangle=\langle\underline{H}^2_1,\ldots,\underline{H}^2_{k-1}\rangle$, and H^2_k is of type t_j. Lettting $\mathcal{C}_{k-1}=\emptyset$, $\mathbf{x}_0\in\mathcal{C}_0=\emptyset$, and $A_1(\emptyset;t_j)=A_1(t_j)$, then, for $1\le k\le\kappa/2$ and any $\mathbf{x}_{k-1}\in\mathcal{C}_{k-1}$, we can count $|A_k(\mathbf{x}_{k-1};t_j)|$ with sufficient accuracy to finish the proof.

In particular, for $1\le k\le\kappa/2$, on the kth step, after having chosen $\mathbf{x}_{k-1}=\langle\underline{H}^2_1,\ldots,\underline{H}^2_{k-1}\rangle\in\mathcal{C}_{k-1}$, from Lemma 4.11, exactly $3(k-1)$ hyper-rows and $3(k-1)$ hyper-columns of $[n]^3_e$ are removed, so that the points of H^2_k are chosen exactly from some $(n-3(k-1))^2$-array depending on the algorithm up until time k. If, by Lemma 4.11, we let $Q(\emptyset)=\emptyset$, and $Q(\mathbf{x}_{k-1})=\{i_1,\ldots,i_{3(k-1)}\}$ for $2\le k\le\kappa/2$, then $Q(\mathbf{x}_{k-1})$ is the set of row and column coordinates of the hyper-rows and hyper-columns that are removed after the $(k-1)$st step. Then, to pick an unbalanced, non-cylindrical, 2-connected graph H^2_k on the kth step , the first point can be chosen in

$$\sum_{\substack{s_1,s_2=1\\ s_1\neq s_2,\ s_1,s_2\notin Q(\mathbf{x}_{k-1})}}^{n} u_{s_1,s_2}$$

ways. Conditional on the first point, if we let

$$\begin{aligned}
\langle a(t_1), b(t_1)\rangle &= \langle 0,0\rangle,\\
\langle a(t_2), b(t_2)\rangle &= \langle 1,1\rangle,\\
\langle a(t_3), b(t_3)\rangle &= \langle 0,1\rangle,\\
\langle a(t_4), b(t_4)\rangle &= \langle 1,0\rangle,
\end{aligned}$$

the second point, for the graph H_k^2 to be of type t_j, can be chosen in

$$\sum_{\substack{s_3=1\\ s_3\neq s_1,s_2,\ s_2\notin Q(\mathbf{x}_{k-1})}}^{n} u_{h_{b(t_j)}(g_{a(t_j)}(s_1,s_2),s_3)}$$

Now, let

$$\begin{aligned}
\delta_n &= \max_{i=1}^{4}\max_{j=0}^{\frac{\kappa}{2}-1}\max_{A\subset[n],|A|=k}\{\sum_{k\in A} m_k^i\}\\
\delta_n' &= 8\max\{\delta_n(\sum_{i=1}^{n} m_i^2)(\sum_{i=1}^{n} m_i), \delta_n(\sum_{i=1}^{n} m_i)^2, \delta_n^2(\sum_{i=1}^{n} m_i^2), \delta_n^2(\sum_{i=1}^{n} m_i)^2, \delta_n^3\}\\
\epsilon_n &= \alpha^2\beta n^3(\frac{1}{\alpha^2\beta}(\frac{1}{n^3}\sum_{i=1}^{n} m_i^2\sum_{j=1}^{n} m_j\sum_{s=1}^{n} m_s)-1)+\delta_n' \quad .
\end{aligned}$$

implying by Lemma 4.13 that $\epsilon_n = o_n(n^3)$. Thus, setting $Q(\mathbf{x}_{k-1}) = Q$ to ease the notation, and noting that for any $1 \le i,j,s \le n$,

$$m_{i,j}m_{h_{b(t_j)}(g_{a(t_j)}(i,j),s)} = m_i^2 m_j m_s \quad \text{or} \quad m_{i,j}m_{h_{b(t_j)}(g_{a(t_j)}(i,j),s)} = m_i m_j^2 m_s,$$

it follows from our observations above that

$$\begin{aligned}
2|A_k(\mathbf{x}_{k-1};t_j)| &= \sum_{\substack{i,j=1\\ i\neq j,\ i,j\notin Q}}^{n} m_{i,j}\sum_{\substack{s=1\\ s\neq i,j,\ s\notin Q}}^{n} m_{h_{b(t_j)}(g_{a(t_j)}(i,j),s)}\\
&= \sum_{\substack{i=1\\ i\notin Q}}^{n} m_i^2\sum_{\substack{j=1\\ j\neq i,\ j\notin Q}}^{n} m_j\sum_{\substack{s=1\\ s\neq i,j,\ s\notin Q}}^{n} m_s\\
&= (\sum_{i=1}^{n} m_i^2-\sum_{i\in Q} m_i^2)(\sum_{j=1}^{n} m_j-\sum_{j\in Q\cup\{i\}} m_j)(\sum_{s=1}^{n} m_s-\sum_{s\in Q\cup\{i,j\}} m_s)
\end{aligned}$$

Obviously,

$$
\begin{aligned}
(\sum_{i=1}^{n} m_i^2 - \sum_{i\in Q} m_i^2)(\sum_{j=1}^{n} m_j - \sum_{j\in Q\cup\{i\}} m_j)(\sum_{s=1}^{n} m_s - \sum_{s\in Q\cup\{i,j\}} m_s) \\
\leq \; & \sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j \sum_{s=1}^{n} m_s \\
= \; & \alpha^2\beta n^3 + (\sum_{i=1}^{n} m_i^2 \sum_{j=1} m_j \sum_{s=1}^{n} m_s - \alpha^2\beta n^3) \\
= \; & \alpha^2\beta n^3 + \alpha^2\beta n^3(\frac{1}{\alpha^2\beta}(\frac{1}{n^3}\sum_{i=1}^{n} m_i^2 \sum_{j=1} m_j \sum_{s=1}^{n} m_s) - 1) \\
\leq \; & \alpha^2\beta n^3 + \epsilon_n \quad .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& (\sum_{i=1}^{n} m_i^2 - \sum_{i\in Q} m_i^2)(\sum_{j=1} m_j - \sum_{j\in Q\cup\{i\}} m_j)(\sum_{s=1}^{n} m_s - \sum_{s\in Q\cup\{i,j\}}^{n} m_s) \\
\geq \; & (\sum_{i=1}^{n} m_i^2 - \delta_n)(\sum_{j=1} m_j - \delta_n)(\sum_{s=1}^{n} m_s - \delta_n) \\
= \; & (\sum_{i=1}^{n} m_i^2 \sum_{j=1} m_j \sum_{s=1}^{n} m_s) - (2\delta_n(\sum_{i=1}^{n} m_i^2)(\sum_{i=1}^{n} m_i) + \delta_n(\sum_{i=1}^{n} m_i)^2 - \delta_n^2(\sum_{i=1}^{n} m_i^2) - 2\delta_n^2(\sum_{i=1}^{n} m_i)^2 + \delta_n^3)
\end{aligned}
$$

However, we see that

$$
|2\delta_n(\sum_{i=1}^{n} m_i^2)(\sum_{i=1}^{n} m_i) + \delta_n(\sum_{i=1}^{n} m_i)^2 - \delta_n^2(\sum_{i=1}^{n} m_i^2) - 2\delta_n^2(\sum_{i=1}^{n} m_i)^2 + \delta_n^3| \leq \delta_n',
$$

which implies that

$$
\begin{aligned}
(\sum_{i=1}^{n} m_i^2 - \sum_{i\in Q} m_i^2)(\sum_{j=1} m_j - \sum_{j\in Q\cup\{i\}} m_j)(\sum_{s=1}^{n} m_s - \sum_{s\in Q\cup\{i,j\}}^{n} m_s) \\
\geq \; & (\sum_{i=1}^{n} m_i^2 \sum_{j=1} m_j \sum_{s=1}^{n} m_s) - \delta_n' \\
= \; & \alpha^2\beta n^3 + \alpha^2\beta n^3(\frac{1}{\alpha^2\beta}(\frac{1}{n^3}\sum_{i=1}^{n} m_i^2 \sum_{j=1} m_j \sum_{s=1}^{n} m_s) - 1) - \delta_n' \\
= \; & \alpha^2\beta n^3 - \epsilon_n \quad .
\end{aligned}
$$

Thus,

$$||A_k(\mathbf{x}_{k-1};\rho)| - \alpha^2\beta\frac{n^3}{2}| \ \leq \ \epsilon_n \quad .$$

It now follows from Lemma 4.12 that

$$\begin{aligned}\frac{1}{(\kappa/2)!}\sum_{\langle G_1,\ldots,G_{\kappa/2}\rangle\in\Xi_L^o} E(\mathbf{Y}_{G_1})\cdots E(\mathbf{Y}_{G_{\kappa/2}}) &= \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2}n^{(3\kappa)/2} + o_n(n^{(3\kappa)/2}) \\ &= \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2}n^{(3\kappa)/2}(1+o_n(1)).\end{aligned}$$

Equation $(**)$

Let $n,\ \kappa\in\mathbb{Z}^+$, and $\sigma\in P^*_{[\kappa]}$. To simplify, let $\mathcal{C}_\sigma = l$, and $r_s^\sigma = r_s$ for $s=1,\ldots,l$. For $s=1,\ldots,l$, let m be the number of r_s that are odd. We are assuming at least one $r_i\neq 2$, which means that the largest l can be is $\frac{\kappa}{2}-\frac{1}{2}$ which is only and integer when κ is odd. Recall also that $\kappa-3m$ has to be even, and $3m\leq\kappa$.

Recall algorithm R from the proof of Lemma 4.10. In this algorithm, if $m=0$, it is understood that steps 3 and 4 are not performed, and if $3m=\kappa$, that steps 1 and 2 are not performed:

1. Step 1: Choose a 2-connected graph.

Let $2\leq i\leq\frac{\kappa-3m}{2}$.

2. Step i: Assuming the 2-connected graphs $G_1^2,\ldots,G_{i-1}^2$ have been chosen mutually disjoint, choose a 2-connected graph from $[n]_e^3\setminus\cup_{j=1}^{i-1}G_j^2$.

3. Step $\frac{\kappa-3m}{2}+1$: Choose a 3-connected graph G_1^3 from $[n]_e^3\setminus\cup_{j=1}^{\frac{\kappa-3m}{2}}G_j^2$.

Let $\frac{\kappa-3m}{2}+1\leq q\leq\frac{\kappa-3m}{2}+m$.

4. Step q: Let $s=q-\frac{\kappa-3m}{2}$. Assuming the 2-connected graphs $G_1^2,\ldots,G_{\frac{\kappa-3m}{2}}^2$, and the 3-connected graphs $G_1^3,\ldots,G_{s-1}^3$ have been chosen mutually disjoint, choose a 3-connected graph G_s^3 from $[n]_e^3\setminus\cup_{i=1}^{\frac{\kappa-3m}{2}}G_i^2\cup\bigcup_{j=1}^{s-1}G_j^3$.

5. Step $\frac{\kappa-3m}{2}+m+1$: Collect the resulting $\frac{\kappa-3m}{2}+m$ connected graphs.

Note that $\sum_{i=1}^{\frac{\kappa-3m}{2}}|G_i^2|+\sum_{j=1}^m|G_j^3| = 2\frac{\kappa-3m}{2}+3m=\kappa\leq|[n]_e^3|$, which implies the algorithm is possible.

Let $0<3m\leq\kappa$. We already saw in Lemma 4.10 that algorithm R gives an upper bound on $|g(\sigma)_n^\kappa|$, given by

$$\frac{1}{2!}\Big(\frac{1}{((\kappa-3m)/2)!}(\lambda n^3)^{(\kappa-3m)/2}\frac{1}{m!}(\lambda n^4)^m\Big) \ \leq \ \frac{1}{2!}\frac{\lambda^\kappa}{(\kappa-3m)/2)!m!}n^{(1/2)(3\kappa-m)}$$

$$\leq \frac{1}{2!}\frac{\lambda^\kappa}{(\kappa-3m)/2)!m!}n^{(1/2)(3\kappa-1)} = o_n(n^{(3\kappa)/2}).$$

Now, let $m = 0$. In this case, κ is even, and the largest l can be is $\frac{\kappa}{2}-1$, which would be when there σ consists of one 4 and $\frac{\kappa}{2}-2$ 2's. Then, by Corollary 4.3, and Lemma 4.9, if follows that for all $n \geq N$,

$$|g(\sigma)_n^\kappa| \leq \lambda n^5(\lambda n^3)^{\frac{\kappa}{2}-2} \leq \lambda^\kappa n^{(3\kappa)/2-1} = o_n(n^{(3\kappa)/2}),$$

λ and N from Lemma 4.9.

This just follows from using the proceeding simple algorithm:

1. Step 1: Choose an 4-connected set G^1.

Let $2 \leq i \leq \frac{\kappa}{2}-1$.

2. Step i: Assuming the connected graphs $G^1, \ldots, G^{i-1}$, $|G^s| = 2$, $2 \leq s \leq i-1$, have been chosen mutually disjoint, choose an 2-connected graph from $[n]_e^3 \setminus \cup_{j=1}^{i-1} G^j$.

3. Step $\frac{\kappa}{2}$: Form the collection of $\frac{\kappa}{2}-1$ connected graphs $\{G^j\}_{j=1}^l$.

As $\sum_{j=1}^j |G^j| = \kappa \leq |[n]_e^3|$, this algorithm is possible for any $n,\ \kappa \in \mathbb{Z}^+,\ \kappa \leq |[n]_e^3|$. Then, any member of $\{\{B(G)_1, \ldots, B(G)_{N(G)}\}\}_{G \in g_n^\kappa(\sigma)}$ can be achieved with this algorithm, because $m = 0$, and thus, we can use Lemma 4.3.

Thus, we have proved that $|g_n^\kappa(\sigma)| = o_n(n^{(3\kappa)/2})$. Because $|Y_{i,j,k}| < C$ for all $(i,j,k) \in \mathbb{Z}_{e,+}^3$, we conclude

$$\sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \leq C^\kappa o_n(n^{(3\kappa)/2}) = o_n(n^{(3\kappa)/2})$$

Q.E.D.

Proposition 2.6 *Let $n,\ \kappa \in \mathbb{Z}^+$, $\kappa \leq |[n]_e^3|$.*

$$\sum_{\sigma \in \mathcal{P}_{[\kappa]}} \sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \frac{t^\kappa}{n^{(3\kappa)/2}} = \begin{cases} \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} t^\kappa(1+o_n(1)) & : \kappa \text{ even} \\ t^\kappa o_n(1) & : \kappa \text{ odd.} \end{cases}$$

Proof. For $\kappa \in \mathbb{Z}^+$, κ even, let $\sigma_2 \in \mathcal{P}_{[\kappa]}^*$ such that $r_s^{\sigma_2} = 2$ for $1 \leq s \leq \kappa/2$. Then, we begin by noting that for $\kappa \geq 1$ and $\sigma \in \mathcal{P}_{[\kappa]}$, we have established by Lemma 4.10 and Lemma 4.14 that

$$\sum_{\{(i_s,j_s,k_s)\}_{s=1}^\kappa \in g_n^\kappa(\sigma)} E(Y_{i_1,j_1,k_1} \cdots Y_{i_\kappa,j_\kappa,k_\kappa})$$

$$= \begin{cases} \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} n^{(3\kappa)/2}(1+o_n(1)) & : \quad \sigma=\sigma_2 \\ o_n(n^{(3\kappa)/2}) & : \quad \text{else} \end{cases}$$

It should be said that $\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})$ is not only $o_n(n^{(3\kappa)/2})$, but actually equal to zero when $\sigma\in\mathcal{P}_{[\kappa]}\backslash\mathcal{P}^*_{[\kappa]}$.

So, we have for κ odd that

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{P}_{[\kappa]}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}} \\
&\qquad = \frac{t^\kappa}{n^{(3\kappa)/2}}\sum_{\sigma\in\mathcal{P}_{[\kappa]}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \\
&\qquad = \frac{t^\kappa}{n^{(3\kappa)/2}}\sum_{\sigma\in\mathcal{P}_{[\kappa]}} o_n(n^{(3\kappa)/2}) \\
&\qquad = t^\kappa o_n(1),
\end{aligned}$$

as $p(\kappa)\le 2^\kappa<\infty$.

Similarly, for κ even,

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{P}_{[\kappa]}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}} \\
&= \sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma_2)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}} \\
&\quad + \sum_{\substack{\sigma\in\mathcal{P}_{[\kappa]}\\ \sigma\ne\sigma_2}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa})\frac{t^\kappa}{n^{(3\kappa)/2}} \\
&= \frac{t^\kappa}{n^{(3\kappa)/2}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma_2)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \\
&\quad + \frac{t^\kappa}{n^{(3\kappa)/2}}\sum_{\substack{\sigma\in\mathcal{P}_{[\kappa]}\\ \sigma\ne\sigma_2}}\sum_{\{(i_s,j_s,k_s)\}_{s=1}^{\kappa}\in g_n^{\kappa}(\sigma)} E(Y_{i_1,j_1,k_1}\cdots Y_{i_\kappa,j_\kappa,k_\kappa}) \\
&= \frac{t^\kappa}{n^{(3\kappa)/2}}\frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} n^{(3\kappa)/2}(1+o_n(1)) + \frac{t^\kappa}{n^{(3\kappa)/2}}\sum_{\substack{\sigma\in\mathcal{P}_{[\kappa]}\\ \sigma\ne\sigma_2}} o_n(n^{(3\kappa)/2}) \\
&= \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} t^\kappa(1+o_n(1)) + t^\kappa o_n(1) \\
&= \frac{1}{(\kappa/2)!}(\frac{\alpha^2\beta(\rho+\tau+\nu+\eta)}{2})^{\kappa/2} t^\kappa(1+o_n(1)).
\end{aligned}$$

again because $p(\kappa)\le 2^\kappa<\infty$.

5 Proof of Lemma for Theorem 1.2

5.1 Proof of Lemma 3.1

Lemma 3.1 *Let* $\{m_i\}_{i=1}^{\infty} \in L$. *Then, for* $1 \le l \le 4$,

$$\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} \xrightarrow{P} \alpha^2 \beta t_l \quad .$$

Proof.
Let $1 \le l \le 4$. First, we note that

$$\begin{aligned}
\lim_n \frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} & E(Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}) \\
&= t_l \lim_n \frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} m_{s_1,s_2} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} \\
&= t_l \lim_n \frac{1}{n^3} \sum_{s_1,s_2=1}^{n} m_{s_1,s_2} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} + \\
&\quad t_l \lim_n \frac{1}{n^3} \sum_{s_1=1}^{n} m_{s_1,s_1} \sum_{\substack{s_3=1 \\ s_3 \neq s_1}}^{n} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} \\
&= t_l \lim_n \frac{1}{n^3} \sum_{s_1,s_2=1}^{n} m_{s_1,s_2} \sum_{s_3=1}^{n} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} + \\
&\quad t_l \lim_n \frac{1}{n^3} \sum_{s_1,s_2=1}^{n} m_{s_1,s_2} \sum_{s_3 \in \{1,2\}} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} + \\
&\quad t_l \lim_n \frac{1}{n^3} \sum_{s_1=1}^{n} m_{s_1,s_1} \sum_{\substack{s_3=1 \\ s_3 \neq s_1}}^{n} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} \\
&\rightarrow \alpha^2 \beta t_l \quad .
\end{aligned}$$

Thus, it suffices to assume that $E(Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}) = 0$ for all $1 \le s_1, s_2, s_3 \le$

n, $1 \leq k_1 \leq m_{i,j}$, $1 \leq k_2 \leq m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}$, and show that

$$\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} \xrightarrow{P} 0 \quad .$$

In consideration thereof, let $\epsilon > 0$. Then,

$$\begin{aligned} &\mathbb{P}(|\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}| > \epsilon) \\ &\quad \leq \frac{1}{\epsilon^2 n^6} E(\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2})^2 \end{aligned}$$

However,

$$\begin{aligned} &E(\frac{1}{n^3} \sum_{\substack{s_1,s_2=1 \\ s_1 \neq s_2}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1,s_2\}}}^{n} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2})^2 \\ \leq\ & \frac{1}{n^6} \sum_{s_1,\ldots,s_6=1}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} \sum_{k_3=1}^{m_{s_4,s_5}} \sum_{k_4=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)}} \\ & |E(Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} Y_{s_4,s_5,k_3} Y_{h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6),k_4})| + \\ \leq\ & \frac{1}{n^6} \sum_{\substack{s_1,\ldots,s_6=1 \\ h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)=\langle s_1,s_2 \rangle}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{k_2=1}^{m_{s_1,s_2}} \sum_{k_3=1}^{m_{s_4,s_5}} \sum_{k_4=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)}} \\ & |E(Y_{s_1,s_2,k_1} Y_{s_1,s_2,k_2} Y_{s_4,s_5,k_3} Y_{h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6),k_4})| + \\ & \frac{1}{n^6} \sum_{\substack{s_1,\ldots,s_6=1 \\ \langle s_4,s_5 \rangle=\langle s_1,s_2 \rangle}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} \sum_{k_3=1}^{m_{s_1,s_2}} \sum_{k_4=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_6)}} \\ & |E(Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} Y_{s_1,s_2,k_3} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_6),k_4})| + \\ & \frac{1}{n^6} \sum_{\substack{s_1,\ldots,s_6=1 \\ h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)=\langle s_1,s_2 \rangle}}^{n} \sum_{k_1=1}^{m_{s_1,s_2}} \sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}} \sum_{k_3=1}^{m_{s_4,s_5}} \sum_{k_4=1}^{m_{s_1,s_2}} \\ & |E(Y_{s_1,s_2,k_1} Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2} Y_{s_4,s_5,k_3} Y_{s_1,s_2,k_4})| + \end{aligned}$$

$$\frac{1}{n^6}\sum_{\substack{s_1,\dots,s_6=1\\ \langle s_4,s_5\rangle=h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}^{n}\sum_{k_1=1}^{m_{s_1,s_2}}\sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}\sum_{k_3=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}\sum_{k_4=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)),s_6)}}$$

$$|E(Y_{s_1,s_2,k_1}Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_3}Y_{h_{b(t_l)}(g_{a(t_l)}(h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)),s_6),k_4})|+$$

$$\frac{1}{n^6}\sum_{\substack{s_1,\dots,s_6=1\\ h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)=h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}^{n}\sum_{k_1=1}^{m_{s_1,s_2}}\sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}\sum_{k_3=1}^{m_{s_4,s_5}}\sum_{k_4=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}$$

$$|E(Y_{s_1,s_2,k_1}Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}Y_{s_4,s_5,k_3}Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_4})|+$$

$$\frac{1}{n^6}\sum_{\substack{s_1,\dots,s_6=1\\ h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)=\langle s_4,s_5\rangle}}^{n}\sum_{k_1=1}^{m_{s_1,s_2}}\sum_{k_2=1}^{m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}}\sum_{k_3=1}^{m_{s_4,s_5}}\sum_{k_4=1}^{m_{s_4,s_5}}+$$

$$|E(Y_{s_1,s_2,k_1}Y_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3),k_2}Y_{s_4,s_5,k_3}Y_{s_4,s_5,k_4})|$$

$$\leq \frac{1}{n^6}\sum_{s_1,s_2,s_4,s_5,s_6=1}^{n} m_{s_1,s_2}^2 m_{s_4,s_5} m_{h_{b(t_l)}(g_{a(t_l)}(s_4,s_5),s_6)}+$$

$$\frac{1}{n^6}\sum_{s_1,s_2,s_3,s_6=1}^{n} m_{s_1,s_2}^2 m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_6)}+$$

$$\frac{1}{n^6}\sum_{s_1,\dots,s_5=1}^{n} m_{s_1,s_2}^2 m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} m_{s_4,s_5}+$$

$$\frac{1}{n^6}\sum_{s_1,s_2,s_3,s_6=1}^{n} m_{s_1,s_2} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}^2 m_{h_{b(t_l)}(g_{a(t_l)}(h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)),s_6)}$$

$$\frac{1}{n^6}\sum_{s_1,\dots,s_5=1}^{n} m_{s_1,s_2} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)}^2 m_{s_4,s_5}+$$

$$\frac{1}{n^6}\sum_{s_1,\dots,s_5=1} m_{s_1,s_2} m_{h_{b(t_l)}(g_{a(t_l)}(s_1,s_2),s_3)} m_{s_4,s_5}^2$$

$$\leq \frac{1}{n^6}\sum_{s_1,s_2,s_4,s_5,s_6=1}^{n} m_{s_1}^2 m_{s_2}^2 m_{s_4}^2 m_{s_5} m_{s_6}+$$

$$\frac{1}{n^6}\sum_{s_1,s_2,s_3,s_6=1}^{n}(m_{s_1}^3 m_{s_2}^3 m_{s_3} m_{s_6}+m_{s_1}^4 m_{s_2}^2 m_{s_3} m_{s_6})$$

$$\frac{1}{n^6}\sum_{s_1,\dots,s_5=1}^{n} m_{s_1}^3 m_{s_2}^2 m_{s_3} m_{s_4} m_{s_5}+$$

$$
\begin{aligned}
& \frac{1}{n^6} \sum_{s_1,s_2,s_3,s_6=1}^{n} (m_{s_1}^4 m_{s_2} m_{s_3}^2 m_{s_6} + m_{s_1}^3 m_{s_2}^2 m_{s_3}^2 m_{s_6} + m_{s_1}^3 m_{s_2} m_{s_3}^3 m_{s_6}) \\
& \frac{1}{n^6} \sum_{s_1,\ldots,s_5=1}^{n} m_{s_1}^2 m_{s_2} m_{s_3}^2 m_{s_4} m_{s_5} + \\
& \frac{1}{n^6} \sum_{s_1,\ldots,s_5=1} m_{s_1}^2 m_{s_2} m_{s_3} m_{s_4}^2 m_{s_5}^2 \\
= \; & \frac{6}{n} \max_{0 \le i_1, j_1, \ldots, i_5, j_5 \le 4} \{(E(X_1)^{i_1})^{j_1} \cdots (E(X_1)^{i_5})^{j_5}\}(1 + o_n(1))
\end{aligned}
$$

Q.E.D.

References

[1] Andrews, G. (1976) *The theory of partitions, Encyclopedia of Mathematics and its Application Volume 2.* Addison-Wesley. Reading, MA.

[2] Stanley, R. (1986) *Enumerative Combinatorics.* Wadsworth. Monterey, CA.