- **Solutions to the Equ'n of Motion of a Lightly Damped, Driven Harmonic Oscillator**

$F_0 = k X_D$ in our demo on the air track

The equ'n of motion is

$$\boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)}$$

The driving angular frequency

Let's look for a late time (or 'particular') sol'n. We'll try

$$\boxed{x(t) = A\cos(\omega t - \delta)}$$

We need to determine A and δ, the amplitude and phase

Taking derivatives

$$\begin{cases} \dfrac{dx}{dt} = -\omega A\sin(\omega t - \delta) \\[2ex] \dfrac{d^2x}{dt^2} = -\omega^2 x(t) \end{cases}$$

The equ'n of motion gives

$$-\omega^2 x(t) + 2\beta(-\omega A\sin(\omega t - \delta)) + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

OR $$(\omega_0^2 - \omega^2)A\cos(\omega t - \delta) - 2\beta\omega A\sin(\omega t - \delta) = \frac{F_0}{m}\cos(\omega t)$$

Using trig identities

WE CAN MATCH LIKE WIGGLES

$$(\omega_0^2-\omega^2)\left[A(\cos(\omega t))\cos\delta + \sin(\omega t)\sin\delta\right]$$

$$- 2\beta\omega A\left(\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta\right) = \frac{F_0}{m}\cos(\omega t)$$

GATHERING TERMS IN $\cos\omega t$ AND $\sin\omega t$, (AND DIVIDING BY A)

$$(\omega_0^2-\omega^2)(\cos\delta + 2\alpha\omega\sin\delta)\cos(\omega t) + ((\omega_0^2-\omega^2)\sin\delta$$

$$- 2\beta\omega\cos\delta]\sin(\omega t) = \frac{F_0}{Am}\cos(\omega t)$$

TO HOLD FOR ALL TIMES, THE COEFFICIENTS OF $\cos(\omega t)$ AND $\sin(\omega t)$ MUST ~~BE~~ SATISFY

$$\begin{cases} (\omega_0^2-\omega^2)\cos\delta + 2\beta\omega\sin\delta = \dfrac{F_0}{Am} & \text{AND} \quad (1.) \\ (\omega_0^2-\omega^2)\sin\delta - 2\beta\omega\cos\delta = 0 & \quad (2.) \end{cases}$$

FROM THE SECOND EQU'N WE HAVE

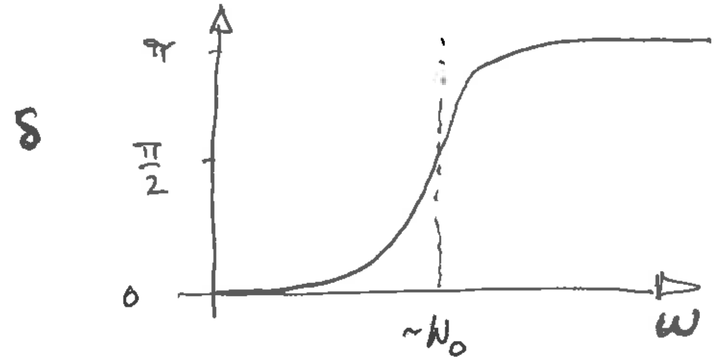
$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

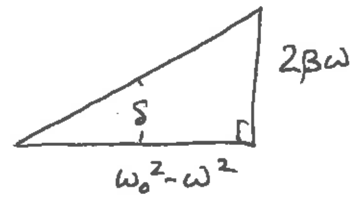
OR

$$\boxed{\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)}$$

THE PHASE

THIS PHASE LOOKS A BIT LIKE THIS





ω DRIVING ANGULAR FREQUENCY

CONTINUING WITH THE SOL'N,

WE CAN SUBSTITUTE (2) INTO (1) TO ELIMINATE $\cos\delta$ AND DO THIS AGAIN TO ELIMINATE $\sin\delta$. THIS ODD MANUVER GIVES

TWO EQU'NS

$$\begin{cases} \left[\dfrac{(\omega_0^2-\omega^2)^2}{2\beta\omega} + 2\beta\omega\right]\sin\delta = \dfrac{F_0}{Am} \\[2ex] \left[(\omega_0^2-\omega^2) + \dfrac{4\beta^2\omega^2}{\omega_0^2-\omega^2}\right]\cos\delta = \dfrac{F_0}{Am} \end{cases}$$

TO ISOLATE A WE CAN COLLECT TERMS

$$\left\{ \left[(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2\right]\left(\frac{1}{2\beta\omega}\right)\sin\delta = \frac{F_0}{Am} \right\} \times 2\beta\omega$$

$$\left\{ \left[(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2\right]\left(\frac{1}{\omega_0^2-\omega^2}\right)\cos\delta = \frac{F_0}{Am} \right\} \times (\omega_0^2-\omega^2)$$

MULTIPLYING THROUGH AND SQUARING (AT LAST THE PHASES GO AWAY VIA $\cos^2\delta + \sin^2\delta = 1$) AND ADDING YIELDS

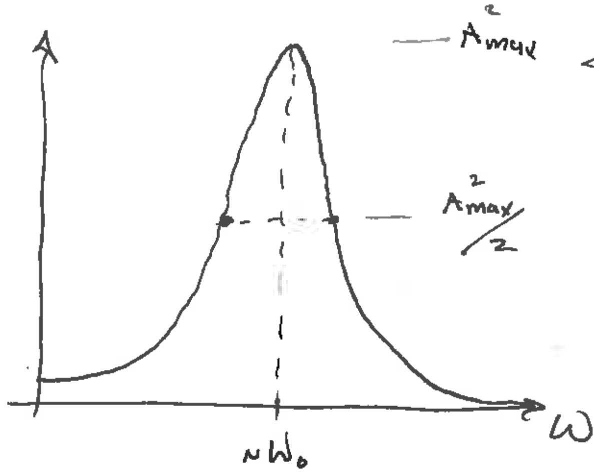
$$\left[(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2\right]^2 = \left(\frac{F_0}{Am}\right)^2 \left(\cancel{(2\beta\omega)^2 + (\omega_0^2-\omega^2)^2}\right)$$

$$\therefore \quad \boxed{A = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}}}$$

THE AMPLITUDE!

SO WE HAVE A LATE TIME SOL'N $x(t) = A\cos(\omega t - \delta)$ WITH A DRIVING FREQUENCY-DEPENDENT AMPLITUDE AND PHASE. THE AMPLITUDE LOOKS A BIT LIKE

WE OFTEN PLOT AMPLITUDE SQUARED $\to A^2$



TO BE MEASURED IN LAB THIS WEEK

$$Q \simeq \frac{\omega_0}{2\beta}$$