Waves and Fields (PHYS 195): Week 4

Spring 2021 1.5

Intro:

This week we finish our study of damped driven oscillators and resonance in these systems and start on our study of waves, beginning with transverse waves on a string. We'll find the equation of motion for waves on a string – the **wave equation***. We will flesh out the study of waves by studying the solutions to the wave equation, including harmonics, standing waves, the principle of superposition, boundary conditions, and Fourier series – but not all this week. We will have one math interlude on partial derivatives.*

Reading:

- Friday: pages 425 - 6 Kleppner and Kolenkow Section 10.3 and HRW 15.6 (The treatment of resonance in HRW is superficial but still worth a read.)
- Monday: HRW 16.1 - 16.3
- Wednesday: HRW 16.4 - 5

Physics Topics:

- Resonance
- The dependence of amplitude and phase of driving angular frequency ω
- Waves - equation of motion
- Transverse waves
- Phasors

Math Topics:

- Partial derivatives
- Wave equation (in 1 dimension)

Problems: Due Tuesday March 2 at 11:59 PM on gradescope code ZR34XK

(1) Equations!
  (a) Write down the equation of motion for a damped harmonic oscillator
  (b) Circle the natural angular frequency in the equation.
  (c) Write the general solution for a lightly damped harmonic oscillator

(2) In the future when you study circuits built from capacitors, inductors, and resistors you will find that the charge q, analogous to x, satisfies the equation of motion

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0,$$

where C is the capacitance, R is the resistance, and L is the inductance. We haven't meant any of these quantities in 190 and 195 but based on what we have studied, what are the natural angular frequency ω_o and β for this system in terms of $L, R,$ and C?

(3) Q, amplitude and phase in driven, damped harmonic motion
  (a) Re-express the amplitude $A(\omega)$ in terms of Q (rather than β) and ω, the driving frequency.

(b) Using Wolfram alpha, mathematica, or another program, plot resonance curves, amplitude squared A^2 vs. driving frequency ω, for lightly damped systems with $Q = 35$ and $Q = 3.5$. It is handy to plot $A^2\omega_o^4/\left(F_o/m\right)^2$ in terms of ω/ω_o so you can plot from from $\omega/\omega_o = 0$ to $\omega/\omega_o = 2$.

(c) Re-express the phase shift $\delta(\omega)$ in terms of Q instead of β.

(d) *Sketch* the phase shifts $\delta(\omega)$ on the same interval of ω for the same Q's as above.

(4) A lightly damped system with a natural angular frequency of $\omega_o = 7.7$ 1/s and a Q of 50 is driven at an angular frequency of 9.0 rad/s. What is the phase shift δ between the driving force and the system?

(5) Phase shift: Using two or three rubber bands and a mug (or other suitable mass) build a lightly damped oscillator. Connect the bands to make a long "spring".

(a) Determine the natural frequency of your oscillator.

(b) Bounce your mug and observe the phase when you drive it below the natural frequency, at the natural frequency, and above the natural frequency. Make a sketch of phase δ versus driving angular frequency ω for your system.

(6) You pilot a spacecraft to a black hole with mass M and enter an orbit. At a radial distance from the planet r, your potential energy is

$$U(r) = U_o\left(-\frac{R}{r} + a^2\frac{R^2}{r^2}\right)$$

where U_o, R, and a are all constants and $0 < r < \infty$. Assume the spacecraft has a mass m.

(a) Find the equilibrium position of the spacecraft.

(b) Find the first three terms of the Taylor series around the stable equilibrium point.

(c) Find k_{eff}.

(d) What is the angular frequency of small oscillations in the radial position of the spacecraft? This means that the spacecraft orbits the black hole at a radius that undergoes simple harmonic motion.

(7) Many modern towers contain huge damped oscillator systems designed to oscillate at the same frequency as the buildings themselves. For instance the Taipei 101 tower has a 728 ton pendulum built into the 90 - 87th floors. You can view a video of the relative motion during an earthquake on this same web page.

(a) Why are these damped oscillator systems built?

(b) In the video the period of oscillation is about 7.1 s. Assuming a lightly damped simple pendulum, find the natural angular frequency.

(c) Suppose that in 10 periods the amplitude of oscillation is reduced from the maximum of 1.4 m to 0.80 m. Find the effective damping coefficient b.

(d) To be most effective at reducing the amplitude of oscillations, what sort of system (lightly damped, critically damped, or overdamped) would you choose?

(8) View the second video showing masses on springs. The display on the function generator is in Hz.

(a) Describe what happens during the video.

(b) The mass on the super-bouncy mass-on-a-spring is 10.0 g. What is the spring constant?

(9) Finding Q: We now have two expressions for the quality factor

$$Q \simeq \frac{\omega_o}{2\beta} \text{ derived from the definition for light damping}$$

and

$$Q = \frac{f_R}{\Delta f} \text{ from lab.}$$

This question relates the two. Throughout this problem we work with lightly damped systems for which $\beta/\omega_o < 1$.

(a) As we have seen,

$$A^2(\omega) = \frac{(F_o/m)^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

By using your calculus powers, locate the maximum and show that the resonant angular frequency is given by

$$\omega_R = \sqrt{\omega_o^2 - 2\beta^2}.$$

(b) Show that for lightly damped systems $\omega_R \simeq \omega_o$.

(c) Now find the maximum amplitude, showing

$$A_{max}^2 \simeq \frac{(F_o/m)^2}{4\beta^2\omega_o^2}.$$

(d) Verify that

$$\frac{A_{max}^2}{2} \text{ occurs when } \omega \simeq \omega_o \pm \beta.$$

(e) Finally, with the full width at half maximum (FWHM) of the amplitude squared curve show that

$$Q \simeq \frac{\omega_o}{2\beta} \simeq \frac{\omega_o}{\Delta\omega} \simeq \frac{f_R}{\Delta f}$$

so that they are the same quantity for light damped oscillators. You might find reviewing K&K pages 426-8 helpful.

(10) *Did we miss a systematic error?!?* In lab last week you had a physical pendulum built from a ring, string, brass sphere, and paperclip.

(a) Find the moment of inertia of just the sphere, I_s. You will need measurements from your lab notebook.[1]

(b) Given your measurements in your lab find the moments of inertia I of the other parts of your pendulum, and the whole pendulum. When I did this I found the table and info on page 274 in HRW helpful.

(c) Compare the different moments of inertia to I_s. Are any of these signifiant?

(d) Using this moment of inertia, find the period of oscillation and compare it to the one you used in lab. Does your measured period agree with this period?

(e) Discuss this potential systematic error and whether you should correction your earlier analysis.

Lab:

No lab due to the wellness day but the next one is Waves on a String: Investigating harmonics, speed of propagation, and other properties of 1D waves. It will be useful to have read 16.12-16.13 before lab.

A look ahead. . .

[1] If you cannot find the data then, after gnashing your teeth in distress, use these numbers: Ring diameter 2.42 cm and mass 6.8 g; String mass 2.6 g; Ball radius 1.27 cm and mass 72.0 g; and paperclip mass 1.24 g.

We move onto harmonics and sound next week. To read ahead see the end of Chapter 16 and the beginning of Chapter 17.