Waves and Fields (PHYS 195): **Review**

Spring 2009 v2.0

Laisssez les bonnes ondes rouler! (Let the good waves roll!)

1. PRE-WAVES:

1.1. **Taylor Expansion.** Any analytic function may be expanded around any point $x_o = x + \Delta x$ as

$$U(x+\Delta x) = U(x) + \frac{dU}{dx}|_x \Delta x + \frac{1}{2}\frac{d^2U}{dx}|_x \Delta x^2 + ... \tag{1}$$

The small angle approximations are derived with this e.g. $\sin\theta \approx \theta$.

1.2. **Equations of Motion.** Newton's

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$$

gives equations of motion. These are second order differential equations (e.g. see equation **??**). For a unique result we need two *initial conditions* per independent variable (e.g. initial position $x(0)$ and initial velocity $v(0)$).

1.3. **Potential Energy.** The potential energy $U(x)$ of a system determines the local and nature of equilibria. The first derivative dU/dx vanishes at equilibria. These points are stable, unstable, or neutral if d^2U/dx^2 is greater than zero, less than zero, or vanishing, respectively.

1.4. **Simple Harmonic Motion (SHM).** Simple harmonic motion is a universal behavior of systems around equilibrium (derived via the potential energy and the Taylor expansiont). As derived from Newtonian mechanics the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2}$$

They are characterized by ω. For a mass on a spring $\omega = \sqrt{k/m}$ while a pendulum has $\omega = \sqrt{g/l}$. The solutions can be written as

$$x(t) = x_o \sin(\omega t + \phi) \tag{3}$$

where x_o and ϕ are fixed by *initial conditions.*

SMO's are great ways to store energy. The total energy of an oscillator is

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{4}$$

SHM occurs in many systems including any linear media[1] like Jello!

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[1] Satisfying

$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{A}$$

where Y is Young's modulus. We saw that jello jiggled with SHM (in this case is was shear motion and Y was replaced by the shear modulus). Volume deformations around equilibrium satisfy

$$\frac{\Delta V}{V} = -\frac{1}{B}\Delta P$$

with B being the bulk modulus. We used this to derive the wave equation for sound.

1.5. **Damped, driven oscillation.** An oscillator with both a velocity-dependent damping force, $-b\mathbf{v}$, and a driving force with angular frequency ω, $f_o\cos(\omega t)$, has the equation of motion

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega_o^2x=\frac{f_o}{m}\cos(\omega t) \tag{5}$$

This has late time solutions

$$x(t)=x_o(\omega)\cos[\omega t+\phi(\omega)] \tag{6}$$

where $x_o(\omega)$ depends on the driving ω

$$x_o(\omega)=\frac{f_o}{m}\frac{1}{\sqrt{(\omega_o^2-\omega^2)^2+(\frac{\omega b}{m})^2}} \tag{7}$$

and the phase $\phi(\omega)$ is

$$\phi(\omega)=\arctan\left[\frac{b\omega}{m(\omega^2-\omega_o^2)}\right] \tag{8}$$

Damped, driven oscillators are characterized by Q, the fractional energy dissipation per radian or $Q\approx m\omega/b$. The characteristic feature of damped, driven oscillators is *resonance* which is the (often catastrophic) growth in amplitude x_o as the driving frequency approaches $\sqrt{\omega_o-b^2/2m}$. Nature has a way to save itself from resonant destruction - *waves.* Waves carry potentially destructive energy away from the oscillator. This is often accomplished by interactions of the system with its environment.

2. FIELDS

Fields are physical and take values at any point in spacetime (x,y,z,t). The subject of fields goes beyond the "point interaction" picture of Newtonian mechanics. Fields accurately model much of known physics. Our typical examples are gravitational, electric and magnetic fields. Fields are visualized with **field lines**.

Motion of a charged particle (q) can be found using

$$F=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{9}$$

The Coulomb force (or Gauss's law) says that the charge of a point charge Q

$$\mathbf{E}=\frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\mathbf{r}$$

Notice that the last two equations give Coulomb's relation: the force between any two point charges is proportional to the product of the charges and $1/r^2$.

The electric potential ("volts") is defined relative to your favorite reference point (e.g. tip of the whale's tail or $r=\infty$). It is the amount of work required to move a unit of charge from the reference point to where the potential is evaluated.

$$\Delta V=V_b-V_a=-\int_a^b\mathbf{E}\cdot d\mathbf{r}$$

Flux e.g.

$$\Phi_E=\int\mathbf{E}\cdot d\mathbf{A}$$

figures prominently Maxwell's equations of E&M. In Gauss's law it is flux around a closed surface. In Faraday and Maxwell's correction term it is around a surface bounded by a closed curve. Flux can be roughly defined as the number of field lines threading through the surface. A non-vanishing time rate of change of the flux can be accomplished **three** ways, by changes in orientation ("changing dot product"), field value, or area.

E&M fields propagate as waves. For waves in the x-direction the fields satisfy

$$\frac{1}{\epsilon_o \mu_o} \frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(similarly for $\mathbf{B}$) in vacuum.

2.1. **E&M devices.** Motors use the $v \times B$ force and the resulting torque $\tau = \mu \times \mathbf{B}$ where μ is the magnetic moment $\mu = I\mathbf{A}$.

Circuits: We constructed devices from resistors (R), capacitors (C), and batteries (V) connected by wires. Kirchhoff's rules say "The sum of the potentials in any loop is equal to zero" and "At each vertex the sum of the currents going in is equal to the sum of the currents going out." These give the equations for current and charge.

3. WAVES

Familiar from observations of water the examples we studied this semester were string, spring, light, sound, gravity, and water waves.

The wave equation for $u(x,t)$ in one spatial dimension is

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \tag{10}$$

It has solutions $u(x,t) = f(kx \pm \omega t)$ where f is any function and the *phase velocity* $v = \omega/k$. We most often consider periodic solutions for instance the right moving wave

$$u(x,t) = u_o \sin(kx - \omega t)$$

Waves are characterized by their frequency (color), polarization (if transverse) and intensity. Waves satisfy the **superposition principle** (since the equation of motion is linear). This gives rise to many, many effects including interference.

3.1. **Geometric Optics.** The "Waves travel in straight lines" approximation. It is good when the devices uses (such as slits) are large compared to the wavelength. The straight lines are rays.

Snell's relation

$$n_1 \sin\theta_i = n_2 \sin\theta_r \tag{11}$$

Mirrors and lenses: Trace principle rays and use algebra in

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \tag{12}$$

Watch out for signs! There are two different types of magnification, angular $m = \theta_i/\theta_o$ and $M = h_i/h_o$. The last is useful when the image and object are not "far away" or at ∞.

3.2. **Physical Optics.** "Waves are wavey" Based on Huygens' principle, the main effects are *interference* and *diffraction.* For instance, for Young's double slit experiment interference between the two sources gives maximum amplitude at

$$d \sin\theta = m\lambda \tag{13}$$

in which d is the slit spacing and $m = 0, \pm 1, \pm 2, \ldots.$ Such effects are seen when the size of the devices used, e.g. slits, lenses, are within an order of magnitude of the wavelength of the wave. Physical optics includes geometric optics as a limit.

A single slit produces a *diffraction* pattern. The location of the dark bands are determined by

$$a \sin\theta = m\lambda$$

where a is the slit spacing and $m = 1, 2, 3, \ldots.$

If the source or observer is moving then we have the Doppler effect which is a shift in frequency

$$f' = f\left(\frac{v \pm v_{obs}}{v \mp v_{source}}\right)$$

3.3. **Wave resonance.** Confined to an interval, waves exhibit resonance (e.g. standing waves) when the interval allows constructive interference between incident and reflected waves.