Waves and Fields (PHYS 195): **Review**

Spring 2017 v1.0

Laisssez les bonnes ondes rouler! (Let the good waves roll!)

Here-in lies a too-brief review of the highlights of the material we discussed this spring...

1. Pre-waves:

1.1. Taylor Expansion and Universality of SHM. Any analytic function - most importantly for us *potential energy* $U(x)$ - may be expanded around any point $x_o = x + \Delta x$ as

$$U(x+\Delta x) = U(x) + \left.\frac{dU}{dx}\right|_x \Delta x + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_x \Delta x^2 + ... \tag{1}$$

The first physically important, non-vanishing term for a system moving in the neighborhood of a stable equilibrium ($dU/dx = 0$) point is the second derivative. This is the effective spring constant for oscillations around this point

$$k_{eff} = \left.\frac{d^2U}{dx^2}\right|_{x=x_o}.$$

The small angle approximations are also derived with this e.g. $\sin\theta \approx \theta$.

1.2. Equations of Motion. Newton's

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$$

gives equations of motion. These are second order differential equations (e.g. see equation 2). For a unique result we need two *initial conditions* per independent variable (e.g. initial position $x(0)$ and initial velocity $v(0)$).

1.3. Potential Energy. The potential energy $U(x)$ of a system determines the local and nature of equilibria. The force is related to the slope of the potential via

$$\vec{F}_x = -\frac{\partial U}{\partial x} \text{ or in general } \vec{F} = -\nabla U.$$

So we can see that the first derivativedU/dx vanishes at equilibria when $F = 0$. These equilibria are stable, unstable, or neutral if d^2U/dx^2 is greater than zero, less than zero, or vanishing, respectively.

1.4. Simple Harmonic Motion (SHM). Simple harmonic motion is a universal behavior of systems around stable equilibria (derived via the potential energy and the Taylor expansion above). We found that the SHM the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2}$$

Motion is characterized by ω or the period $T = 2\pi/\omega$. For a mass on a spring $\omega = \sqrt{k/m}$ while a pendulum has $\omega = \sqrt{g/l}$. The solutions of the equation of the motion can be written as

$$x(t) = x_m \sin(\omega t + \varphi) \tag{3}$$

(cosine works just as well) where the *amplitude* x_o and *phase* φ are fixed by *initial conditions*.

These solutions satisfy *superposition* so we can add them. (This property is a result of the linearity of the equation of motion.)

SMO's are a great way to store energy. The total energy of an oscillator is

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_o^2 = \frac{1}{2}m\omega^2 x_o^2 \tag{4}$$

SHM occurs in many systems including any linear media[1] like Jello! By the universality of SHM argument we found (as above)

$$k_{eff} := \left.\frac{d^2U}{dx^2}\right|_{x_o}$$

when x_o is a stable equilibrium point. If you know the potential energy this is the most direct way to obtain the "frequency of small oscillations".

1.5. **Damped oscillation.** A "damped" oscillator has a linear velocity-dependent damping force, $\mathbf{F}_d = -b\mathbf{v}$ has the equation of motion

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_o^2 x = 0 \tag{5}$$

We used $\beta = b/(2m)$ so the equation of motion becomes

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_o^2 x = 0 \tag{6}$$

For lightly damped cases ($\beta/\omega_o \ll 1$), this equation of motion has solutions

$$x(t) = x_m e^{-\beta t}\sin(\omega t + \varphi).$$

While x_m and phase φ are set by initial conditions, $\omega = \sqrt{\omega_o^2 - \beta^2}$. The now-time dependent amplitude, $x_m e^{-\beta t}$, is the envelope into which the oscillations, $\sin(\omega t + \varphi)$, are fit.

1.6. **Damped, driven oscillation.** An oscillator with both a velocity-dependent damping force, $-b\mathbf{v}$, and a driving force with angular frequency ω, $f_o\cos(\omega t)$, has the equation of motion

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_o^2 x = \frac{f_o}{m}\cos(\omega t) \tag{7}$$

We used $\beta = b/(2m)$ to characterize the solutions. For *lightly damped motion* ($\beta/\omega_o \ll 1$), this equation of motion has late time solutions

$$x(t) = A(\omega)\cos[\omega t + \phi(\omega)] \tag{8}$$

where the *amplitude* $A(\omega)$ depends on the driving ω

$$A(\omega) = \frac{f_o}{m}\frac{1}{\sqrt{(\omega_o^2 - \omega^2)^2 + (\frac{\omega b}{m})^2}} \tag{9}$$

and the *phase* $\phi(\omega)$ is

$$\phi(\omega) = \arctan\left[\frac{b\omega}{m(\omega^2 - \omega_o^2)}\right] \tag{10}$$

---

[1]Satisfying

$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{A}$$

where Y is Young's modulus. We saw that jello jiggled with SHM (in this case is was shear motion and Y was replaced by the shear modulus). Volume deformations around equilibrium satisfy

$$\frac{\Delta V}{V} = -\frac{1}{B}\Delta P$$

with B being the bulk modulus. We used this to derive the wave equation for sound.

Damped, driven oscillators are characterized by the *quality factor* $\boldsymbol{Q}$, the fractional energy dissipation per radian or $Q \approx m\omega/b$. The characteristic feature of damped, driven oscillators is *resonance* which is the (often catastrophic) growth in amplitude x_o as the driving frequency approaches $\sqrt{\omega_o - b^2/2m}$.

Nature has a way to save itself from resonant destruction - *waves*. Waves carry potentially destructive energy away from the oscillator. This is often accomplished by interactions of the system with its environment. We derived the resulting *wave equation* (see section 3) for waves on a string and for sound. Light also satisfies the wave equation; light is a wave.

2. Fields

Fields are physical and take values at any point in spacetime (x, y, z, t). The subject of fields goes beyond the "contact interaction" picture of Newtonian mechanics. Fields accurately model much of known physics. Our typical examples are gravitational, electric and magnetic fields. Fields are visualized with **field lines**.

2.1. Electro- and Magneto-statics.

Electric fields are produced from electric charges. The electric field of a point charge Q is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}$$

Notice that this equations and $\mathbf{F} = q\mathbf{E}$ give Coulomb's relation: the force between any two point charges is proportional to the product of the charges and $1/r^2$. Two opposite charges at a fixed distance d apart form a dipole with moment qd. Immersed in an electric field the dipole enjoys a torque $\vec{\tau} = \mathbf{p} \times \mathbf{E}$.

The electric potential V in ("volts") is defined relative to your favorite reference point (e.g. tip of the whale's tail or $r = \infty$). It is the amount of work required to move a unit of charge from the reference point to where the potential is evaluated.

$$\Delta V = V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{r}. \text{ So for a single charge } V(r) = \frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

which gives us the easiest method of finding electric fields - add up all the electric potentials.

Magnetic fields are sourced by *moving* charges, such as current flow I. From Biot-Savart's Law,

$$\vec{B} = \frac{\mu_o}{4\pi}\int \frac{I d\vec{\ell} \times \hat{r}}{r^2}$$

Motion of a charged particle (q) can be found using

$$F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{11}$$

Motors use the $v \times B$ force and the resulting torque

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

where $\vec{\mu}$ is the magnetic moment $\mu = I\vec{A}$.

As it turns out - and we didn't study this in detail - E&M fields propagate as waves. For waves in the x-direction the fields satisfy

$$c^2\frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(similarly for $\mathbf{B}$) in vacuum, where $c = \sqrt{1/\epsilon_o\mu_o}$, the speed of light, about 3×10^8 m/s.

3. Waves

Familiar from observations of water the examples we studied this semester were string, spring, light, sound, gravity, and water waves. The *wave equation* for $u(x,t)$ in one spatial dimension is

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \tag{12}$$

It has solutions $u(x,t) = f(kx \pm \omega t)$ where f is any function and the *phase velocity* $v = \omega/k$. We most often consider periodic solutions for instance the right moving wave

$$u(x,t) = u_o \sin(kx - \omega t)$$

Energy transport for a wave is quantified by intensity the "energy flow per unit area". For waves on a string $I = 2\pi^2 \mu v f^2 A^2$.

Waves are characterized by their frequency (color), polarization (if transverse) and intensity. Waves satisfy the **superposition principle** (since the equation of motion is linear). This gives rise to many, many effects including interference, standing waves, and diffraction.

3.1. Doppler Shifts.

If the source and/or observer is moving then we have the Doppler effect which is a shift in frequency. For *sound* we have

$$f' = f\left(\frac{v \pm v_{obs}}{v \mp v_{source}}\right) \tag{13}$$

For *light* the relation can only depend on the relative velocity and so it is simpler

$$f' = f\sqrt{\frac{c \pm v}{c \mp v}} \tag{14}$$

3.2. Geometric Optics.

The "Waves travel in straight lines" approximation. It is good when the devices uses (such as slits) are large compared to the wavelength. The straight lines are rays.

Snell's relation

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{15}$$

Mirrors and lenses: Trace principle rays and use algebra in

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \tag{16}$$

Watch out for signs!

3.3. Physical Optics.

"Waves are wavey" Based on Huygens' principle, the main effects are *interference* and *diffraction*. For instance, for Young's double slit experiment interference between the two sources gives maximum amplitude at

$$d \sin\theta = n\lambda \tag{17}$$

in which d is the slit spacing and $n = 0, \pm1, \pm2, \ldots$. Such effects are seen when the size of the devices used, e.g. slits, lenses, are within an order of magnitude of the wavelength of the wave. Physical optics includes geometric optics as a limit.

A single slit produces a *diffraction* pattern. The location of the dark bands are determined by

$$a \sin\theta = m\lambda \tag{18}$$

where a is the slit spacing and $m = 1, 2, 3, \ldots$. **Phasors** provide a way to find the intensity. We use them to find the amplitude, then we square them. They incorporate both interference and diffraction. For example we found that the double slit intensity is

$$I = I_o \left[\frac{\sin\left(\frac{\pi a \sin\theta}{\lambda}\right)}{\frac{\pi a \sin\theta}{\lambda}}\right]^2 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \tag{19}$$

3.4. **Wave resonance.** Confined to an interval, waves exhibit resonance (e.g. standing waves) when the interval allows constructive interference between incident and reflected waves.

4. Experimental Methods

There are many techniques you learned in lab, e.g. how to accurately time the period of pendulum - many periods is better! When taking measurements there are:

4.1. **Significant figures.** "Sig figs" allow us to track the precision of our quantitative work.

4.2. **Uncertainties.** : To propagate uncertainties first look for dominant error and then add all *independent* uncertainties in quadrature. For addition and subtraction

$$z = x + y \text{ or } z = x - y \text{ then } \delta z = \sqrt{\delta x^2 + \delta y^2} \tag{20}$$

For multiplication and division then add the relative uncertainties in quadrature,

$$z = xy \text{ or } z = x/y \text{ then } \frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2} \tag{21}$$

For a power, multiply the relative uncertainty by the power, i.e. if

$$z = x^n \text{ then } \frac{\delta z}{z} = n\frac{\delta x}{x}. \tag{22}$$

In general for a calculated quantity $q = q(x, ..., z)$ then

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + ... + \left(\frac{\partial q}{\partial x}\delta z\right)^2} \tag{23}$$