Solutions:

(1) The $B$-field at the center of ring is $\frac{\mu_o I}{2r}$ (see equation 6.54), or $B = \frac{\mu I}{a}$ here since $r = a/2$. (Some of you used the result for a solenoid, which is the same up to numerical coefficients.)

The total stored energy is then

$$U = \frac{1}{2} \mu_o \left( \frac{\mu_o I}{a} \right)^2 (\pi a^2 \cdot a) = \frac{\mu_o \pi a I^2}{2}$$

The power dissipated is through the resistor ($R = \pi / a \sigma$) to the time to decay is roughly

$$\tau \approx \frac{U}{I^2 R} = \frac{\mu_o \pi a I^2}{2} \frac{1}{I^2 R} \approx \mu_o a^2 \sigma$$

For Earth’s core and $a \approx 3000 \text{ km}$ and $\sigma \approx 10^6 \text{ (Ω m)}^{-1}$ we would have

$$\tau \approx 10^{13} \text{ s}, \text{ or } 3000 \text{ centuries},$$

a long time for us and a short time for the Earth.

Some of you worked this by finding the inductance of a solenoid. That is also fine and yields similar results. Your numerical results may differ from these since you might have retained the numerical factors.

(2) This one is done in the book, but that did not stop the questions! Here’s a couple things that came up in our discussions. (1) A low pass filter “passes” or let’s through an alternating signal with a low frequency and blocks high frequency signals. (2) As described in the problem the circuit is just a simple loop of the signal generator, resistor, and capacitor, which explains how the solution gets started. If you wanted to allow a little current through into the amplifier on the right you could solve for the resulting system by using Kirchhoff’s rules. This is done in the last part of the solution for the better low pass filter designs.

(3) There’s a solution in the book - and we did this problem in class!

(4) The loop equation is

$$RI + \frac{Q}{c} = E_o \cos \omega t$$

Complexifying and using the integral to relate $\tilde{Q}$ and $\tilde{I}$ gives

$$RI e^{i\omega t} + \frac{\tilde{I}}{i \omega C} e^{i\omega t} = E_o e^{i\omega t} \implies \tilde{I} = \frac{E_o}{R + 1/i \omega C} = \frac{E_o (R + i/\omega C)}{R^2 + 1/\omega^2 C^2}$$

Writing this in polar form gives

$$\tilde{I} = \frac{E_o}{\sqrt{R^2 + 1/\omega^2 C^2}} e^{\phi} \text{ with } \phi = 1/R \omega C.$$ 

The actual current is the real part of this

$$I = \frac{E_o}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t - \phi).$$

For large driving frequencies the current goes to $E_o/R$ and the phase goes to zero. The oscillations are too fast for charge to build up on the capacitor so it essentially isn’t there. For small driving frequencies the amplitude goes to zero and the phase goes to $\pi/2$. Now the current
is very small so it is like not having the resistor in the circuit. The phase is just $\pi/2$ for a capacitor in series with a signal generator.

(5) We need the impedance for the circuit. With all the elements in parallel we have

$$\frac{1}{Z} = \frac{1}{R} - \frac{i}{\omega L} + i\omega C,$$

the total admittance (= 1/Z). With the numbers given,

$$Z \simeq 15.7 + i124 \, \Omega$$

The imaginary part is dominant so the inductance is roughly equal to $i\omega L$. The contribution of the capacitor is small. For the high driving frequency the inductance is

$$Z \simeq 1.01 - i31.8 \, \Omega$$

Now the inductor is negligible. The frequency is high enough so that the capacitor lets current through easily compared to the resistor and inductor.

To find the frequency for which the impedance is a maximum we need the magnitude,

$$|Z| = \frac{1}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}}.$$

This is large when the denominator is small; when the last factor vanishes or when

$$\omega = \frac{1}{\sqrt{LC}} \simeq 10^{6} \, s^{-1},$$

or when $f \simeq 160 \, Hz$. This gives $|Z|_{\text{max}}$ of 1000 $\Omega$. This frequency effectively removes the capacitor and inductor as the have equal and opposite currents.

(6) (The driving voltage is 10.1 V, which is not very clear.) The circuit is just a loop with the four elements connected in series. Given the angular frequency $\omega = 2\pi 1000 \, s^{-1}$ and $V_o = I_o|Z|$ for the capacitor

$$I_o \simeq 0.0974 \simeq 0.1 \, A.$$

Since this is a series circuit, this is the current through everything. Computing the impedance for the whole circuit,

$$I_o = \frac{V_o}{|Z|} \implies 0.0974 = \frac{10.1}{\sqrt{35^2 + (\omega L - 1/\omega C)^2}}.$$

Solving this puppy for $L$ picks up two roots 0.041 and 0.0098 H. From these we have voltage drops of $I_o \omega L = 25.1$ or 6.0 V. If we measure 25.4 V then the second root is ruled out and the first agrees reasonably well with the first root.

(7) Inside the capacitor we have $E = \sigma/\epsilon_o = Q/A\epsilon_o$. The rate of change of the flux is thus,

$$A \frac{dQ}{dt} A\epsilon_o = \frac{I}{\epsilon_o} \text{ and } I_d = I$$

as expected. As for the sign,... As for the electric field it points to the left, since the right plate is positive. But the positive current flows onto the left plate, making the derivative $\partial E/\partial t$ point to the right. So this is how the displacement current flows to the right and into the bag-like surface. Closing the surface so that the total surface is $S$ and $S'$ (or what I called $S$ and $S_{B}$ in class), and has no boundary, we see then that the displacement current flows in with magnitude $I$ and the actual current flows out with magnitude $I$. Hence, the net flux vanishes as it must, since the line integral of $B$ around no boundary vanishes.
(8) We’re going to take the radius of that wee hole, \( a \) in the large sphere (of radius \( R \)) to be really small: \( a \ll R \). This was one of the confusing parts to this problem, but it helps us compute the flux and the magnetic field. Of course, near the wire it looks like a long wire so that \( B = \mu_o I / (2\pi a) \) where \( a \) is small compared the radius of the sphere. The LHS of Maxwell’s equation is then

\[
\oint \mathbf{B} \cdot d\mathbf{s} = (\mu_o I / (2\pi a))(2\pi a) = \mu_o I.
\]

Now, the RHS side of Maxwell’s equation. We have a point charge so \( E = Q / (4\pi \epsilon_o r^2) \). At the radius of the sphere \( R \), the flux is

\[
\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{4\pi \epsilon_o R^2} (4\pi R^2) = \frac{Q}{\epsilon_o}
\]

(Of course I neglected the bit of flux through the wee hole.) Hence, the RHS is

\[
\mu_o \epsilon_o \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{a} = \mu_o \epsilon_o \frac{\partial}{\partial t} \frac{Q}{\epsilon_o} = \mu_o I
\]

as expected; LHS = RHS!

(9) Tricky problem conceptually. Not so bad ‘calculationally’. The idea is to compare two cases, one as drawn and one where the charge has moved a distance \( vdt \) closer. The difference in flux through the two disks is equal to the flux through the cylindrical surface of radius \( r \) and height \( vdt \),

\[
d\Phi_E = \mathbf{E} \cdot d\mathbf{a} = 2\pi r vdt E \cos(\pi/2 - \theta) = 2\pi r vdt E \sin(\theta)
\]

where the trig works out as shown.

Maxwell’s equation is satisfied if

\[
\oint \mathbf{B} \cdot d\mathbf{a} = \frac{1}{c^2} \frac{d\Phi_E}{dt} \Rightarrow 2\pi r B = \frac{1}{c^2} 2\pi r vdt E \sin(\theta) \quad \text{or} \quad \mathbf{B} = \frac{v}{c^2} \times \mathbf{E}
\]

which is as we hoped.

(10) Question from Roger Easton’s talk (worth 1/10 points)