Solutions:

(1) By linearity of the field we can consider this the sum of a sphere with charge density ρ and radius a and a sphere with charge density $-\rho$ and radius $a/2$. For both points we need the field at the surface of a sphere. Using Gauss's law and a spherical gaussian surface just at r,

$$E4\pi r^2 = \frac{\rho}{\epsilon_o}\frac{4}{3}\pi r^3 \implies E = \frac{\rho r}{3\epsilon_o}.$$

Now at A,

$$E_{total} = E_{big\,sphere} + E_{little\,sphere} = 0 - \frac{\rho a}{2\cdot 3\epsilon_o} = -\frac{\rho a}{6\epsilon_o}.$$

The field points upwards.

At B we need the field of the little sphere a away from the surface ($r = 3a/2$) so we need to use Gauss's law again

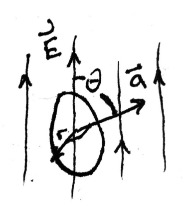
$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q_{\text{encl}}}{\epsilon_o} \text{ gives } E4\pi\left(\frac{3a}{2}\right)^2 = -\frac{\rho}{\epsilon_o}\frac{4}{3}\pi\left(\frac{a}{2}\right)^3 \implies \mathbf{E} = -\frac{\rho a}{54\epsilon_o}\hat{r}.$$

That's the bit for the small sphere. Now adding the two fields gives,

$$E = \frac{\rho a}{3\epsilon_o} - \frac{\rho a}{54\epsilon_o} = \frac{17\rho a}{54\epsilon_o}.$$

This points downwards.

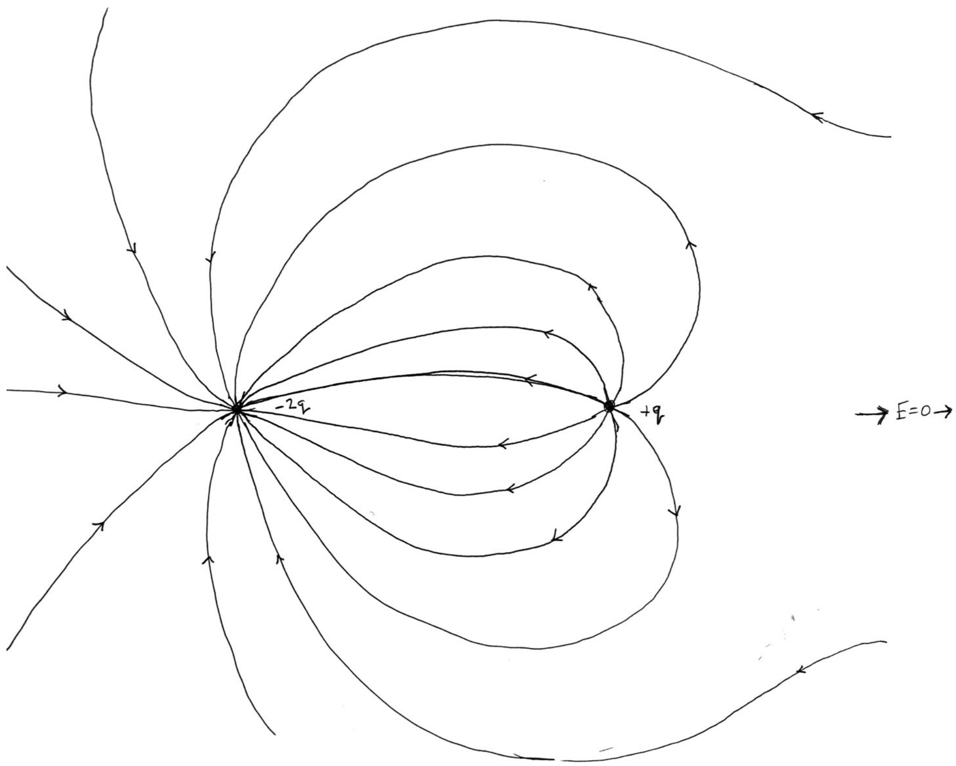
(2) Here's my sketch of the geometry



where $\theta = 60°$ and $r = 5$ cm. The flux is

$$\Phi = \vec{E}\cdot\vec{a} = Ea\cos\theta = E\pi r^2\cos\theta \simeq 0.59\ \text{Nm}^2/\text{C}$$

(3) Here's my sketch of the field:



The point where the electric field vanishes is $1 + \sqrt{2} \simeq 2.4$ to the right of the $+q$ charge (in units of the distance between charges). This point is off the page to the right. Also, there should be 8 lines going out to infinity, since from far away ($r \gg d$) the configuration looks like a point charge of $-q$. I have 7 - sorry!

(4) This is an energy conservation problem. For a spherical object the scalar potential is the same as a point source outside the sphere,

$$V = \frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

(assuming we set $V = 0$ at $r \to \infty$). Signs can be confusing in this problem. I assume David's σ and q have the same sign. Otherwise the "$-q$" charge wouldn't be attracted to the sphere. This means that $-qQ$ and $-q\sigma$ are both negative.

Inside a hollow sphere the electric field vanishes and so the potential is constant. Since the potential is constant, the speed inside will be as well. Now far away, where the potential vanishes, the charge starts from rest so the total energy $\mathcal{E} = 0$. As the charge falls inward falls we have

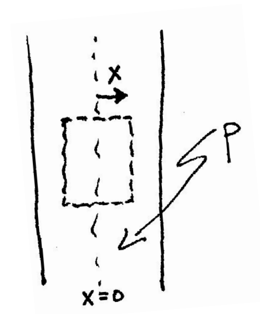
$$\mathcal{E} = 0 = \frac{1}{2}mv^2 + \frac{qQ}{4\pi\epsilon_o}\frac{1}{r}.$$

When $r = R$ we have

$$\frac{1}{2}mv^2 - \frac{qQ}{4\pi\epsilon_o}\frac{1}{R} = 0 \implies v = \sqrt{\frac{2R\sigma q}{\epsilon_o m}}$$

where I have used $Q = 4\pi\sigma R^2$. The velocity is directed inward. This is the same velocity as at the center.

(5) We'll use superposition to find the field. We know from class that the field on either side of the sheet of charge with surface charge density σ has magnitude $E = \sigma/2\epsilon_o$ and points away from the sheet. In the slab the electric field can only point in the x (horizontal) direction so we can use Gauss's law inside. Here's a sketch of the gaussian pillbox and coordinate



Let the area of the ends of the pill box be A then from Gauss' law,

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{encl}}}{\epsilon_o} \text{ we have } 2EA = \frac{\rho}{\epsilon_o} 2xA \implies \mathbf{E} = \frac{\rho x}{\epsilon_o}\hat{\imath}$$

I have called the area of the left and right sides of the gaussian pillbox A. Similarly outside we have

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{encl}}}{\epsilon_o} \text{ or } 2EA = \frac{\rho}{\epsilon_o} dA \implies \mathbf{E} = \frac{\rho d}{2\epsilon_o}\hat{\imath}.$$

This gives rise to three regions for the total electric field. Left of the sheet of charge we add the fields from the sheet and the slab - they both point to the left -

$$\mathbf{E} = -\left(\frac{\sigma}{2\epsilon_o} + \frac{\rho d}{2\epsilon_o}\right)\hat{\imath}.$$

Right of the sheet of charge and in the slab we have

$$\mathbf{E} = \left(\frac{\sigma}{2\epsilon_o} + \frac{\rho x}{\epsilon_o}\right)\hat{\imath},$$

keeping in mind that $x < 0$ in part of this region and that the electric field from the sheet points right. And to the right of everything we have

$$\mathbf{E} = \left(\frac{\sigma}{2\epsilon_o} + \frac{\rho d}{2\epsilon_o}\right)\hat{\imath}.$$

(6) (2 pts.) The charge density varies with radius so we need to integrate to find the amount of charge within a Bohr radius a_o. But first, since we don't know C, **we need to find the constant** C. We know the total charge of the electron is $-e$. Hence,

$$-e = \int \rho dv = -C \int_0^\infty e^{-2r/a_o} 4\pi r^2 dr \text{ or } e = C4\pi 2 \left(\frac{a_o}{2}\right)^3$$

where I looked up the integral on my handy page of integrals – you may wish to start your own collection if you haven't already done so! – I found $C = e/\pi a_o^3$.

The amount of charge q_{a_o} of the electron inside a Bohr radius is then

$$q_{a_o} = -\frac{e}{\pi a_o^3} \int_0^{a_o} e^{-2r/a_o} 4\pi r^2 dr.$$

This is essentially the same integration as before but now the upper limit is at a finite a_o. The integral works out to be

$$q_{a_o} = q\left(-1 + \frac{5}{e^2}\right) \simeq -0.323e \simeq -5.2 \times 10^{-20}\ \mathrm{C},$$

where in the first step I temporarily switched $e \to q$ to distinguish the exponential "$e = \exp(1)$" from the charge "e". The **net** charge is then $q_{net} \simeq 0.677e \simeq 1.08 \times 10^{-19}$ C since there is a proton at the center of the atom.

We're now asked to find the electric field at this radius. The electric field is spherically symmetric (since ρ has no angular dependence) and we can use Gauss's law

$$E 4\pi a_o^2 = \frac{q_{encl}}{\epsilon_o} \simeq \frac{0.677e}{\epsilon_o}.$$

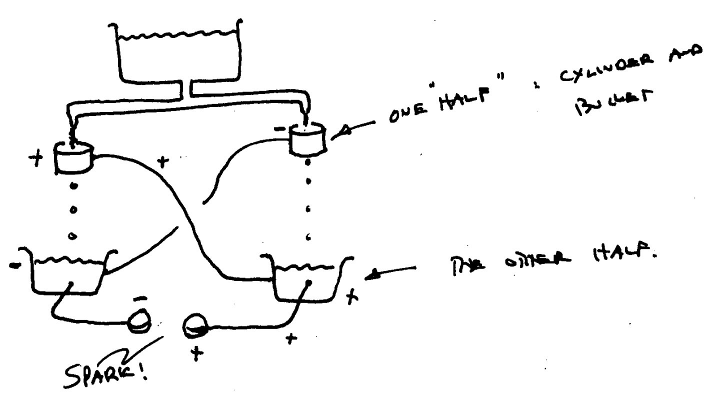
The electric field is

$$\vec{E} \simeq \frac{0.677e}{4\pi\epsilon_o a_o^2}\hat{r} \simeq \left(3.48 \times 10^{11}\ \mathrm{V\ m^{-1}}\right)\hat{r}.$$

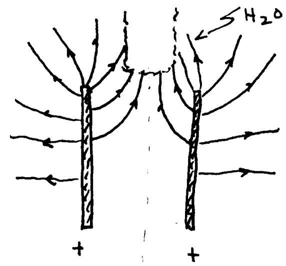
That is a strong field! (If you just computed the electric field due to the electron, would obtain -1.7×10^{11}.)

(7) This one has a solution in the book. Note the use of limits - it's a really good idea! Optional for next time: How would you solve this if the ring was not fixed?

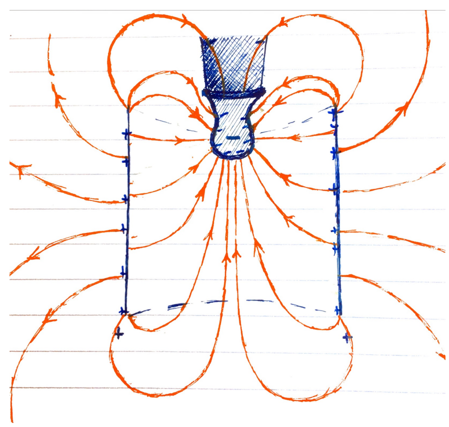
(8) Kelvin Water Dropper: Since a spark is produced when there is a large buildup of charge, the dropper must generate charge on the brass spheres.



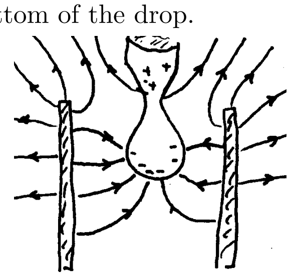
How does this happen? Let's assume an initial asymmetry of charge between the two copper cylinder-bucket halves. (It doesn't matter how small this is.) The cylinders have electric fields like this



The **E**-field causes the OH^- and H^+ ions to migrate in the water. Here's the field



(Thanks to Mikel Zemborain for this sketch!) When a drop forms the effect is greater and negative charge collects at the bottom of the drop.



When the drop pinches off it will be negatively charged. It falls into the bucket connected to the other cylinder, which is already negatively charged, increasing its charge. So the process builds up charge. When the buildup of charge is great enough the spark occurs and the process starts again.

This is a playful example of a physical instability.