Electromagnetism (PHYS 295): Solutions 3

Spring 2024 v1.5

Solutions:

(1) Solved in the text.

(2) (a) Calculating

$$(\nabla \cdot \mathbf{A})_{\text{cartesian}} = 2$$

Great. For the cylindrical case,

$$(\nabla \cdot \mathbf{A})_{\text{cylindrical}} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} = \frac{1}{r}\frac{\partial r^2}{\partial r} = 2.$$

So it works as it must.

(b) As last time the cartesian divergence is simple; 3 in this case. Now setting up the cylindrical calculation

$$\mathbf{A} = x\hat{\imath} + 2y\hat{\jmath} = r\cos\theta(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + 2r\sin\theta(\sin\theta + \cos\theta\hat{\theta})$$

...chugg-chugg...

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta)}{\partial \theta} = 3$$

as it should.

(3) We solve this by first solving for the electric field with Gauss' law inside and outside the cylinder. Since this is a 'long' cylinder, the electric field will point away from the charge and along the radial direction. We'll have gaussian surfaces of radius r and length ℓ inside and outside. Inside, we have $r < R$ and

$$\oint \vec{E} \cdot d\vec{a} = |\vec{E}|2\pi r\ell$$

while

$$Q_{encl} = \int \rho d^3x = \int_0^\ell dz \int_0^{2\pi} d\varphi \int_0^r r \cdot br^2 dr = \ell \cdot 2\pi \cdot \frac{br^4}{4} = \frac{\pi b\ell r^4}{2}$$

so by Gauss' law we have

$$|\vec{E}| = \frac{br^3}{4\epsilon_o}$$

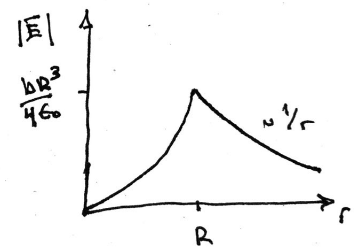
inside. Outside, the total the whole cylinder is enclosed so Gauss' law becomes

$$\oint \vec{E} \cdot d\vec{a} = |\vec{E}|2\pi r\ell = \frac{Q_{encl}}{\epsilon_o} = \ell \cdot 2\pi \cdot \frac{bR^4}{4\epsilon_o}$$

so that

$$|\vec{E}| = \frac{bR^4}{4\epsilon_o r}.$$

(Notice the expressions for the electric field inside and outside match when $r = R$.) Here's a sketch of the magnitude of the electric field



For the potential let's integrate. Inside,

$$\Delta V = V(r) - V(0) = -\int_0^r \vec{E}\cdot d\vec{s} = -\int_0^r |\vec{E}|\cdot dr = -\frac{br^4}{16\epsilon_o}.$$

I'm leaving $V(0)$ unspecified for the moment, anticipating a choice of where we should set $V = 0$. Meanwhile on the outside,

$$\Delta V|_{out} = V(r) - V(R) = -\int_R^r \vec{E}\cdot d\vec{s} = -\int_R^r \frac{bR^4}{4\epsilon_o r}dr = -\frac{bR^4}{4\epsilon_o}\ln\left(\frac{r}{R}\right).$$

Given that the log vanishes when $r = R$ (and diverges at infinity), one nice choice of the $V = 0$ surface is on the surface of the cylinder. This means fixing

$$V(0) = \frac{bR^4}{16\epsilon_o}$$

so that inside

$$V(r) = \frac{bR^4}{16\epsilon_o} - \frac{br^4}{16\epsilon_o} \text{ when } r \le R$$

and outside

$$V(r) = -\frac{bR^4}{4\epsilon_o}\ln\left(\frac{r}{R}\right) \text{ when } r \ge R$$

(4) Solved in the text.

(5) I'll grade these, if there are solutions

(6) For this mathematical exercise we get to practice integrals along paths. First we have a two part path. The first one vanishes since $E_x = 0$ when $y = 0$. For the second part of the path

$$\int \mathbf{E}\cdot d\mathbf{s} = \int_0^{y_1} E_y dy = 3x_1^2 y_1 - y_1^3$$

where $d\mathbf{s} = dy\hat{\jmath}$. The second route also yields this, as it must. So the electric potential

$$\phi(x_1, y_1) = -3x_1^2 y_1 + y_1^3 + C$$

Taking (minus) the divergence returns the electric field.

(7) The proton gains a kinetic energy of eV_{VdG} in the Van de Graaff. If we assume a spherical charge distribution for the silver nucleus then the proton will come to rest again when

$$eV_{VdG} = \frac{47e^2}{4\pi\epsilon_o}\frac{1}{r} \implies r = \frac{47e}{4\pi\epsilon_o V_{VdG}} \simeq 1.3\times 10^{-14}\text{ m } = 120\text{ pm}.$$

From my point of view this is uncomfortably close (and less than!) to the expected 160 pm radius of a silver nucleus. Better modeling of the nucleus would make sense here. Ignoring this, however, we can find the electric field, at this radius of closest approach.

$$|E| = \frac{47e}{4\pi\epsilon_o}\frac{1}{r^2} = \frac{4\pi\epsilon_o}{47e}V_{VdG}^2 \simeq 4 \times 10^{20} \text{ N/C}$$

and this gives an acceleration of

$$a = \frac{eE}{m_p} \simeq 3.8 \times 10^{28} \text{ m/s}^2$$

which is large!

(8) Calculating the divergences

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + 1 - 2 = 0,$$

$$\nabla \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} = 0$$

and

$$\nabla \cdot \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2x + 2x = 4x.$$

(9) Solved in the text.