Solutions:

(1) The expansion is proportional to

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} = \left(\frac{1}{r}\right)\left(\frac{d}{r}\cos\theta + \frac{d^3}{2^2r^3}P_3(\cos\theta) + \frac{d^5}{2^4r^5}P_5(\cos\theta)\dots\right).$$

The first question asks when is it ok to drop the second (and higher order) terms. The Legendre polynomials have a maximum value of 1 so to find the rough distances we can set them to 1. More precisely the questions asks when is

$$1\% \simeq \frac{\frac{d^3}{2^2r^3}}{\frac{d}{r}+\frac{d^3}{2^2r^3}} \simeq \frac{d^2}{4r^2}?$$

Well, when $r = \sqrt{100/4}\ell = 5d$, or 5 times the spacing between the charges. If we keep the octupole term but drop the next one then we are asking at what radius does

$$1\% \simeq \frac{\frac{d^5}{2^4r^5}}{\frac{d}{r}+\frac{d^3}{2^2r^3}+\frac{d^5}{2^4r^5}}?$$

This occurs when

$$\frac{1}{100} \simeq \frac{d^4}{2^4r^4} \text{ or}$$

when

$$r = \sqrt[4]{\frac{100}{2^4}}d \simeq 1.6d$$

which is much closer.

(2) On the z axis

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{r-(\ell/2)^2} - \frac{1}{r+(\ell/2)^2}\right)\hat{\mathbf{k}}$$

Now expanding the fractions

$$\mathbf{E} \simeq \frac{q}{4\pi\epsilon_o}\left(1+\frac{\ell}{r}-1+\frac{\ell}{r}\right)\hat{\mathbf{k}} = \frac{q\ell}{2\pi\epsilon_o r^3}\hat{\mathbf{k}} = \mathbf{E}(r,0),$$

remembering that the dipole moment $p = q\ell$ and $\hat{r} = \hat{\mathbf{k}}$ on the z axis. On the x-axis we have to add in a sine term since the vertical components add while the horizontal components cancel,

$$\mathbf{E} = 2\frac{q}{4\pi\epsilon_o}\left(\frac{1}{r^2+(\ell/2)^2}\right)\left(\frac{\ell}{2\sqrt{r^2+(\ell/2)^2}}\right)(-\hat{\mathbf{k}}).$$

The second factor is the sine written in terms of the coordinates. Expanding the resulting $(r^2+(\ell/2)^2)$ denominator simply gives

$$\mathbf{E} \simeq \frac{q\ell}{4\pi\epsilon_o r^3}(-\hat{\mathbf{k}}) = \mathbf{E}(r,\pi/2),$$

since $\hat{\theta}(\pi/2) = -\hat{\mathbf{k}}$.

(3) To determine whether the function f is a solution to Laplace's equation,

$$\nabla^2 f = 0,$$

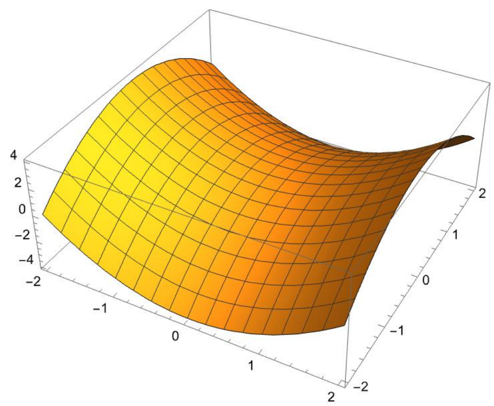
we need the form of the Laplacian ∇^2. In 2D Cartesian coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

so we have

$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(x^2 + y^2) = \frac{d}{dx}(2x) + \frac{d}{dy}(2y) = 4 \neq 0.$$

This is not a solution to Laplace's equation. But $x^2 - y^2$ is a solution since now we have $2 - 2 = 0$ in the last step above. Plotting this as a surface plot gives

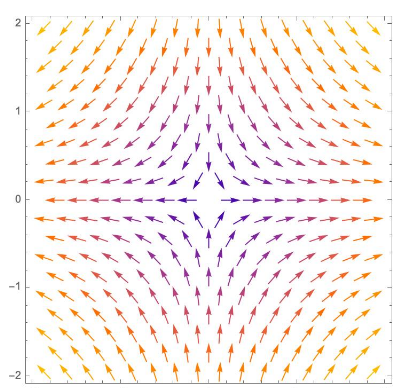


The x-axis is on the lower right.

The gradient is

$$\nabla f = 2x\hat{\imath} - 2y\hat{\jmath}.$$

At (0,1) this is $-2\hat{\jmath}$; at (1,0) this is $2\hat{\imath}$; at (0,-1) this is $2\hat{\jmath}$; and at (-1,0) this is $-2\hat{\imath}$. Here's the vector plot



(4) (2 pts.) Finding E and σ from a potential

(a) To check whether ϕ is a viable potential in a region without charge we must show that $\nabla^2\phi$ vanishes. The given potential has dependence on x and z only so we only need to consider derivatives with respect to x and z

$$\frac{\partial\phi}{\partial x} = -k\,\phi_o e^{-kz}\sin(kx) \text{ and } \frac{\partial\phi}{\partial z} = -k\,\phi_o e^{-kz}\cos(kx).$$

The second derivatives are

$$\frac{\partial^2\phi}{\partial x^2} = -k^2\,\phi_o e^{-kz}\cos(kx) \text{ and } \frac{\partial^2\phi}{\partial z^2} = +k^2\,\phi_o e^{-kz}\cos(kx).$$

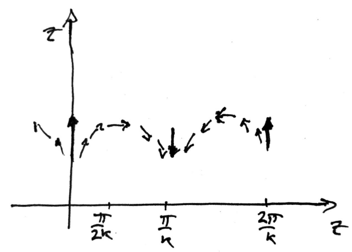
so it is clear that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

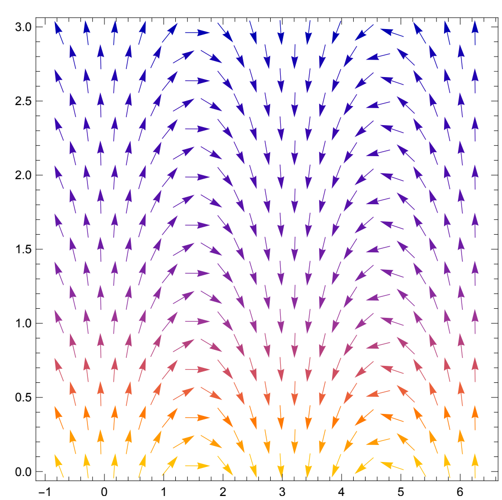
(b) The electric field is given by the gradient so

$$\mathbf{E} = -\nabla\phi = k\,\phi_o e^{-kz}\sin(kx)\,ihat + k\,\phi_o e^{-kz}\cos(kx)\hat{\mathbf{k}} = k\phi\left(\tan kx\hat{\imath} + \hat{\mathbf{k}}\right).$$

So the slope is infinite when $x = 0$, $x = 2\pi/k$, ... and zero when $x = \pi/k$, ... It looks something like



Asking mathematica to plot the vector field gives



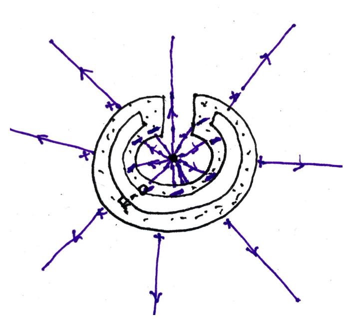
where the magnitude is plotted as a "heat map" with red being large and blue being small. I have set $k = 1$ for this plot.

(c) We can find the charge distribution via the change in the electric field. Given that the charge is only on the sheet at $z = 0$ the field must be symmetric around this surface. Choosing a wee gaussian pillbox with cross-section area A spanning the surface we have, from Gauss's law, $2AE_z = \sigma A/\epsilon_o$ so

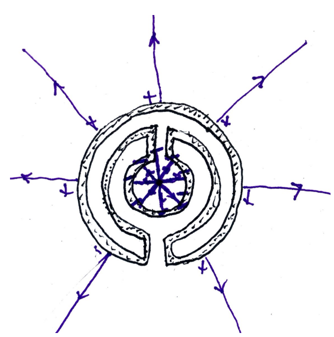
$$\sigma = 2\epsilon_o E_z|_{z=0} = -2\epsilon_o \left.\frac{\partial \phi}{\partial z}\right|_{z=0} = 2\epsilon_o k\, \phi_o \cos(kx).$$

Weird but ok.

(5) In the first configuration, charge of $-q$ accumulates on the inner surface of the inner shell, since it is attracted by the charge q in the center. The approximate charge density is $\sigma \sim -q/(4\pi r^2)$ where r is the radius of the inner surface. The conductor is neutral so a charge of q settles on the outer surface, as far as it can get from the inside charge. (Don't be tempted to add charge to the outer surface of the inner shell and the inner surface of the outer shell. That will yield non-vanishing electric field in the inside cavity in the conductor where the electric field vanishes.) A little bit of charge accumulates in the slot to ensure that $E = 0$ in the shell, but this is less important. Here's a sketch of the charge and field



The second diagram is similar, with $-q$ charge on innermost surface and $+q$ on the outer surface. However, in this case the charge is inside a cavity inside the conductor so the electric is non-vanishing there. Here's a sketch



We are perhaps tempted to add charge on the other surfaces but this would add an electric field inside the conductor.

(6) For the "four-square" configuration with charges at $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$ the potential is proportional to

$$\phi \propto \left[(x-1)^2 + (y-1)^2\right]^{-1/2} - \left[(x+1)^2 + (y-1)^2\right]^{-1/2} + \left[(x+1)^2 + (y+1)^2\right]^{-1/2} - \left[(x-1)^2 + (y+1)^2\right]^{-1/2}$$

(Adding in a z coordinate would not change the behavior.) This is constant on the surface $y = 0$ as we can see from

$$\left[(x-1)^2+1\right]^{-1/2} - \left[(x+1)^2+1\right]^{-1/2} + \left[(x+1)^2+1\right]^{-1/2} - \left[(x-1)^2+1\right]^{-1/2} = 0.$$

Similarly for the $x = 0$ surface. We have now known that this image charge configuration has the correct potentials for the problem of a charged plate bent 90 degrees. Therefore the field will look like