Electromagnetism (PHYS 295): Solutions 5

Spring 2024 v1.0

Solutions:

(1) The approximate parallel plate capacitor has capacitance,

$$C = \frac{\epsilon_o A}{d}.$$

where $A = 1.3$ cm^2. As the air breaks down the electric field reaches $E_! = 3 \times 10^{-6}$ N/C. The field is constant in a parallel plate capacitor so the potential is $V = Ed$. Now, the charge is

$$Q = CV = \frac{\epsilon_o A}{d} Ed = \epsilon_o AE \simeq 4.5 \times 10^{-9} \text{ C}$$

Not so much! (about 2.8×10^{10} electrons.)

(2) Done in text. But (d) is essentially "the image charge configuration works as shown in parts (a) - (c)".

(3) Done in the text. The key step in these problems is to identify which quantity is the same and which quantity adds. For example for capacitors in series the Q is the same and the potentials add.

(4) Done in text but David's version seems more confusing than it needs to be. Here's essentially the same solution but in a form that is easier to follow (maybe!).

We want to maximize the energy stored without exceeding a maximum electric field E_{max}, which the book calls E_0. From a sketch of the charged configuration is clear that the maximum electric field must be on the inner sphere of radius b. It is

$$E_{max} = \frac{Q}{4\pi\epsilon_o} \frac{1}{b^2}$$

This relation links Q and b together. The energy stored is

$$U = \frac{\epsilon_o}{2} \int E^2 d^3x.$$

Anywhere inside the electric field is,

$$E = \frac{Q}{4\pi\epsilon_o} \frac{1}{r^2} = E_{max} \frac{b^2}{r^2},$$

using E_{max} and thus getting the correct scaling of the electric field and the radius b. So,

$$U = \frac{\epsilon_o}{2} \int E_{max}^2 \frac{b^4}{r^4} d^3x = \frac{\epsilon_o E_{max}^2}{2} \int \frac{b^4}{r^4} r^2 dr \sin\theta d\theta d\varphi$$

using the spherical volume element $r^2 \sin\theta d\theta d\varphi dr$. The angular integrations are simple - just giving 4π leaving

$$U = 2\pi\epsilon_o E_{max}^2 \int_b^a \frac{b^4}{r^2} dr$$

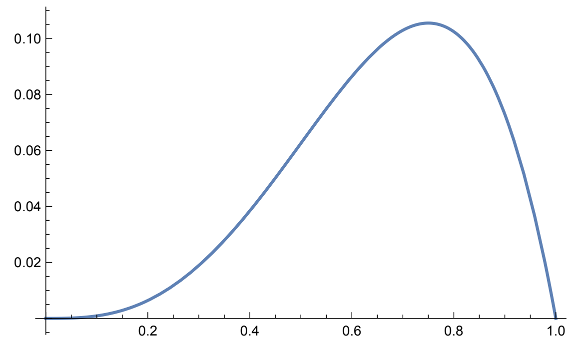
to integrate. This evaluates to

$$U = 2\pi\epsilon_o E_{max}^2 (b^4) \left(-\frac{1}{a} + \frac{1}{b} \right) = 2\pi\epsilon_o E_{max}^2 \left(\frac{1}{b^3} - \frac{b^4}{a} \right). \quad (1)$$

Since we want to store the maximum amount of energy we should extremize this with respect to the inner shell radius b. So

$$\frac{dU}{db} = 0 \implies 3b^2 - 4\frac{b^3}{a} = 0 \text{ or } b = \frac{3}{4}a.$$

Let's check that this is actually a maximum. We can see this with a plot of $U/(2\pi\epsilon_o E_{max}^2 a^3)$ vs. b/a



which peaks at 0.75. (You can also check this by taking another derivative of 1 and seeing that $dU^2/db^2 < 0$ at $b = 3a/4$.) The total energy in the optimal configuration is then U with this value for b, which is

$$U = \frac{27}{128}\pi\epsilon_o E_{max}^2 a^3.$$

Anyone see a reason why "3/4" is natural? (I don't but am curious.)

(5) We have an inner conductor with radius b and charge Q and an outer conductor with radius a and charge $-Q$. The cylinder has length L. The electric field will be radial. I'll choose a cylindrical gaussian surface at radius r. By Gauss' law, E will be

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{encl}}{\epsilon_o} \implies 2\pi r L E = \frac{Q}{\epsilon_o} \text{ or } \mathbf{E} = \frac{Q}{2\pi\epsilon_o L r}\hat{r}$$

I have neglected the fringing field on the ends of the bottle. The magnitude of the potential difference is then

$$V = \int \mathbf{E} \cdot d\mathbf{s} = \int_b^a \frac{Q}{2\pi\epsilon_o L r} dr = \frac{Q}{2\pi\epsilon_o L}\ln(a/b)$$

So the capacitance would be

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_o L}{\ln(a/b)}.$$

For the limit we need to send the radii to 'very large' but keep them separated (or else we have no capacitor!). So let $a = b + d$ where d is the separation of the plates. The log becomes

$$\ln(a/b) = \ln\left(\frac{b+d}{b}\right) = \ln\left(1 + \frac{d}{b}\right) \simeq \frac{d}{b}.$$

This last approximation is a handy one - you can derive it from the Taylor series of $\ln(1+x)$ for small x. Thus,

$$C = \frac{2\pi\epsilon_o L}{\ln(a/b)} \simeq \frac{2\pi b L \epsilon_o}{d} = \frac{\epsilon_o A}{d}$$

as expected. (The area of the 'very large' cylinder or plate is $A \simeq 2\pi b L$.)

(6) So this pile of e^-, each with charge -1.6×10^{-19} C travel roughly at c around the circle of 240 m. This means that the current is effectively the charge times the frequency of revolution

$$I = Qf = 10^{11} \cdot (-1.6 \times 10^{-19}) \cdot \frac{3 \times 10^8}{240} \simeq -0.02 \text{ A} = -20 \text{ mA}.$$

a macroscopic current! The sign shows that the current flows in the opposite direction to the electrons.