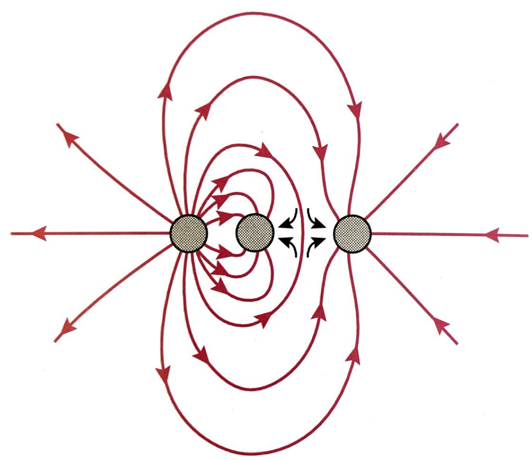
Phys 295: Example Mid-term Solution Sketches

This a mix of hints and more complete solutions.

Problem hints:

(1) Here's a sketch of an electric field created by three charges:



Determine the signs of the charges and their relative magnitudes. (The black curvy lines show the direction of field lines at that point. Why do you think the author decided to include them?)

I'll number the charges from the left q_1, q_2, q_3.

- From the directions of field lines $q_1 > 0$. It is the largest charge.
- From the field lines q_2 is negative. By counting field lines

$$q_2 = \frac{6}{14} q_1$$

- From the field lines q_3 is negative. By counting field lines

$$q_2 = \frac{7}{14} q_1 = \frac{1}{2} q_1$$

The black curves are around an impossible field line; it is inconsistently oriented. So the black curves show what happens to the field lines at this point. They diverge as shown. It is an alternative to a ”twist the star” approach we have used, where we would avoid lines passing through this point.

(2) Arranging charges: There are many ways to do this problem.

(a) I'll place the two $+4q$ charges a distance $2a$ apart and find where the electric field vanishes. This will be where to place the $-1q$ charge. Let the x axis be between the two charges and the origin in the middle. Then

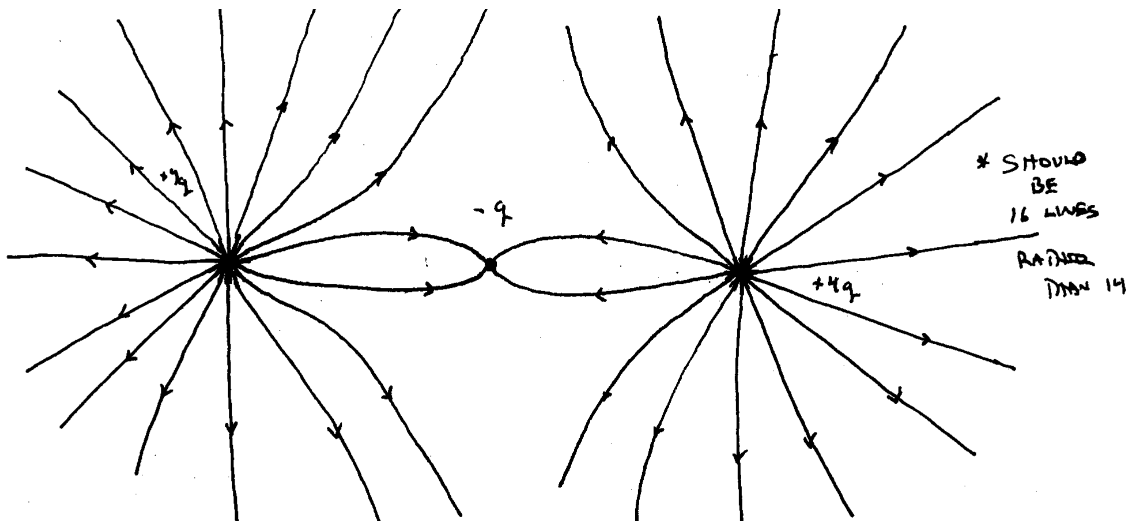
$$\mathbf{E} = \frac{4q}{4\pi\epsilon_o (a+x)^2}\hat{\imath} + \frac{4q}{4\pi\epsilon_o (a-x)^2}(-\hat{\imath}) = -\frac{4axq}{\pi\epsilon_o (x^2-a^2)^2}\hat{\imath}$$

This vanishes at $x = 0$ so - no surprise - the equilibrium is in the center. To check whether the $+4q$ charges are also in equilibrium, consider the one on the positive side of the axis

$$\mathbf{E} = \frac{4q}{4\pi\epsilon_o(2a)^2}\hat{\imath} + \frac{q}{4\pi\epsilon_o a^2}(-\hat{\imath}) = 0$$

so these charge is in equilibrium as well. By symmetry the other $+4q$ charge is as well.

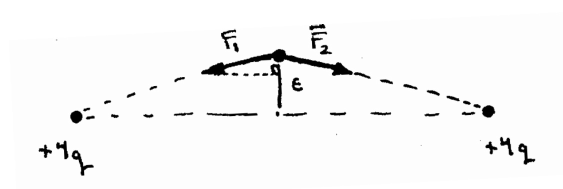
(b) A sketch of field lines



(c) To check the stability of the $-q$ charge I'll displace it slightly in the positive x direction. Then the force on it is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_o}\left[\frac{4q^2}{(a-x)^2}\hat{\imath} - \frac{4q^2}{(a+x)^2}\hat{\imath}\right] = \frac{q^2}{\pi\epsilon_o}\frac{4ax}{(a^2-x^2)^2} > 0$$

so when the charge is displaced to the right it experiences a force to the right; the charge accelerates away from the equilibrium. This is an unstable equilibrium point.
We also need to look at the perpendicular direction. Displacing the charge "upwards" as shown



we have

$$\mathbf{F} = -2\sin\theta\hat{\jmath} = \frac{-8q^2}{4\pi\epsilon_o(a^2+\epsilon^2)}\frac{\epsilon}{\sqrt{a^2+\epsilon^2}}\hat{\jmath} \simeq -\frac{2q^2}{\pi\epsilon_o a^3}\epsilon\hat{\jmath}.$$

This is of the form "$F = -kx$" so the solution is stable in this direction. (The approximation uses the fact that the displacement is small so we can neglect terms of higher order in ϵ/a.) In summary the equilibrium is stable in the perpendicular direction and unstable in along the axis. This is called a "saddle point".

(3) Starting from the differential form of Gauss' law derive the integral form.

Starting with the differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$

we integrate over volume. On the left hand side the divergence theorem gives $\int \mathbf{E} \cdot d\mathbf{a}$ while on the right hand side we obtain the charge enclosed so we obtain the integral form

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{encl}}{\epsilon_o}$$

(4) A solid sphere of radius R is charged with radially dependent density $\rho = k_o r^2$ where k_o is a positive constant in units of C/m^5. Find the $\mathbf{E}$-field inside and outside the sphere.

Hint: Use Gauss's law to show that inside

$$E = \frac{k_o r^3}{5\epsilon_o} \text{ and outside } E = \frac{k_o R^5}{5\epsilon_o r^2}$$

(5) Consider a uniformly charged, long cylinder of radius R. Assume the volume charge density is ρ.
  (a) Using Gauss' law find the electric field inside $(0 \leq r \leq R)$ and outside $(r \geq R)$.
  (b) Sketch the electric field lines.
  (c) Make a plot of $|\mathbf{E}(r)|$ from $r = 0$ to $r = 4R$.

Hint: Inside choosing a gaussian surface of length ℓ and radius $r < R$ Gauss' law gives

$$\int E \cdot da = E \cdot 2\pi r\ell = \frac{1}{\epsilon_o}\rho\pi r^2 \ell \text{ or } E = \frac{\rho r}{2\epsilon_o}$$

Outside is similar but the charge integral is cutoff at R so

$$E = \frac{\rho R^2}{2\epsilon_o r}.$$

(6) An Infinite sheet has a uniform surface charge density σ. Find the $\mathbf{E}$-field everywhere. Solve for the $\mathbf{E}$-field around two sheets of opposite charge separated by a distance d.

Use Gauss's law to show that $E = \sigma/2\epsilon_o$ on both sides. Inside the capacitor the field is σ/ϵ_o. Outside, the field vanishes.

(7) Find the electric potential $V(z)$ on the axis of a disk with uniform surface charge density σ and radius a. Find an expression for the E- field when $z \gg a$ and check that it conforms to what you expect.

By direct integration of

$$dV = \frac{1}{4\pi\epsilon_o}\frac{dq}{r} = \frac{1}{4\pi\epsilon_o}\frac{\sigma r' d\varphi dr'}{r}$$

we find

$$\phi = \frac{\sigma}{2\epsilon_o}\left(\sqrt{z^2 + a^2} - z\right)$$

To check this result I will expand ϕ when $z \gg a$ using $(1+x)^n \simeq 1 + nx$. This far away the disk should look like a point charge. (You might have first found E.) The potential becomes

$$V = \frac{\sigma\, z}{2\epsilon_o}\left(\sqrt{1+\frac{a^2}{z^2}} - 1\right) \simeq \frac{\sigma}{2\epsilon_o}\left(\frac{a^2}{2z^2}\right) = \frac{Q}{4\pi\epsilon_o}\frac{1}{z}$$

with $Q = \sigma\pi a^2$. This yields an electric field of the form $E = Q/4\pi\epsilon_o z^2$ which has the correct form of a point charge.

(8) Find the electric field of a charge q positioned a distance h above a a conducting plate. Find the surface charge distribution on the plate.

An image charge problem. The potential is

$$V = \frac{q}{4\pi\epsilon_o}\left(\left[x^2+y^2+(z-h)^2\right]^{-1/2} - \left[x^2+y^2+(z+h)^2\right]^{-1/2}\right)$$

which is constant on the xy plane. Since $E = \sigma/\epsilon_o$ above a charged conductor and since $E_z = -\partial\phi/\partial z$ we have

$$\sigma = \frac{-qh}{2\pi(x^2+y^2+h^2)^{3/2}}$$

(9) A charge q sits a bit off center by a distance a in a parallel plate capacitor with separation d $(a < d/2)$. Find an approximate form of electric potential in the capacitor. Make a careful sketch of the electric field inside.

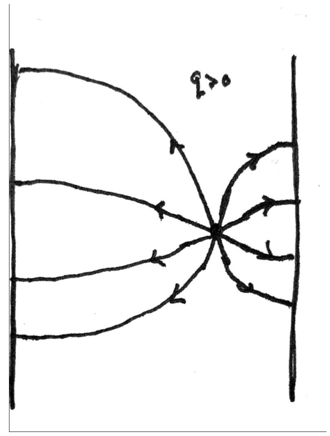
Looks like an image charge problem. Let me set coordinates so that the origin is at the center of the capacitor and next to the charge. Let's assume that the charge is at $x = a$. To start let's place $-q$ charges at $d/2 - a$ beyond the near plate (at $x = d - a$) and $d/2 + a$ beyond the far plate (at $x = -d - a$). Hmm, but each of these will mess with the field at the other plate so let's try to cancel these effects by placing two additional $+q$ charges at $x = 2d + a$ and $x = -2d + a$. The process continues. It would be hopeless except as the image charges are placed further away we can neglect their contribution. The potential is

$$V = \frac{q}{4\pi\epsilon_o}\left(\left[(x-a)^2+y^2\right]^{-1/2} - \left[(x-d+a)^2+y^2\right]^{-1/2} - \left[(x+d+a)^2+y^2\right]^{-1/2}\ldots\right)$$

(To be in 3D you can add in a $+z^2$ in each of the distances.) At $x = \pm d/2$ the potential terms cancel pairwise - but one term is always left over, hence the need for the infinite sum, which can be written as (I think)

$$V = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{(x-a)^2+y^2}} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{(x-nd+a)^2+y^2}} + \frac{(-1)^n}{\sqrt{(x+nd+a)^2+y^2}}\right)\right]$$

Here's my sketch of the field:



(10) A hollow spherical shell of radius a has charge Q, which is uniformly painted over the surface. By finding the energy stored in the electric field, find the energy stored in this configuration.

The energy is

$$U = \frac{\epsilon_o}{2} \int E^2 dv$$

while the field is

$$E = \frac{Q}{4\pi\epsilon_o} \frac{1}{r^2} \text{ for } r > a$$

Thus,

$$U = \frac{\epsilon_o}{2} \int_a^\infty \frac{Q^2}{(4\pi\epsilon_o)^2} \frac{1}{r^4} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_o} \left[-\frac{1}{r} \right]_a^\infty = \frac{Q^2}{8\pi\epsilon_o} \frac{1}{a}$$

(11) Find the monopole, dipole (and quadrupole if you wish) moments of your configuration of charges in problem 2.

For the monopole contribution we add up all the charges to find $Q = 4q - q + 4q = 7q$. The potential associated to this is

$$V = \frac{1}{4\pi\epsilon_o} \frac{7q}{r}$$

where r is from the $-q$ charge at the center.

The dipole term is found from suming over all the qd's for the charges. So the total in the x direction is

$$p = (4q)(-a) + (-q)(0) + (4q)(a) = 0.$$

The dipole moment vanishes so there is no dipole contribution to the potential. (The other directions also vanish because the charges are at $y = 0$ and $z = 0$.)

The quadrupole contribution is a bit more complicated, as it is a two component object - a matrix! The charges are at $x = -a$, $x = 0$, and $x = a$.

$$Q_{xx} = \frac{1}{2} \left(3a^2 - a^2\right) (4q + 4q) = 8qa^2$$

while

$$Q_{yy} = Q_{zz} = \frac{1}{2}(-a^2)(8q) = -4qa^2$$

All the other components vanish. The associated potential is

$$V = \frac{1}{4\pi\epsilon_o}\frac{\sum \hat{r}^i\hat{r}^j Q_{ij}}{r^3} = \frac{1}{4\pi\epsilon_o}\frac{8qa^2 - 8qa^2}{r^3} = 0$$

So, the only contribution to the potential is the monopole term.

(12) A battery, capacitor, and resistor are connected in series around a loop. Initially the there is no charge on the capacitor. What is the charge at time t?

Everything is in series so we have

$$\mathcal{E} + IR + \frac{Q}{C} = 0 \text{ or } \frac{\mathcal{E}}{R} + \frac{dQ}{dt} + \frac{Q}{RC} = 0$$

which has a solution

$$Q(t) = \mathcal{E}C\left(e^{-t/RC} - 1\right)$$

when the capacitor carries no initial charge.