1. Braket (Dirac) Notation

Dirac introduced a very beautiful way of expressing the vectors used in quantum mechanics. This is a short introduction to "braket notation" from the point of view of vector calculus. For those of a more mathematical bent or wanting something more complete see sections 6-20 of Dirac's book, *The Principles of Quantum Mechanics*. Few can match the logical clarity of this work.

Basic idea of a ket: A "ket" $|\cdot\rangle$ is a vector. However, the components may be complex, hence the space of kets is a *complex vector space*. To keep track of *which* vector we add a label. For instance, $|+n\rangle$ might represent the "spin up in the nth direction" vector. Since (suitable) functions can be seen to form a vector space, wavefunctions are also written this same way, e.g. $|\psi\rangle$.

Slogan: "Put what you know in the ket." In quantum mechanics, you identify the vector by the last measurement on the system. Suppose you have a particle in a box. If you observed the particle on the left hand side, say $0 < x < L/2$, then the ket would be $|0 < x < L/2\rangle$.

Basis: If you have N basis vectors $|i\rangle, i = 1, 2, \ldots, N$ then any vector $|v\rangle$ is written as

$$|v\rangle = \sum_i v_i \, |i\rangle.$$

It could also be arranged in a column

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_N \end{pmatrix}$$

These expressions are the analog of the usual

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

in the usual 3D vector space.

Bra: A "bra" $\langle\cdot|$ is "dual" to a vector which means that the *adjoint* of the vector,

$$\langle a| = (|a\rangle)^\dagger.$$

By the dagger $\dagger$ is the usual notation for adjoint. The mechanics of the adjoint take kets to bras and the components to their complex conjugates. For instance, if

$$|neatket\rangle = \frac{i}{\sqrt{2}}\,|3\rangle$$

then

$$\langle neatbra| \equiv (|neatket\rangle)^\dagger = \frac{-i}{\sqrt{2}}\langle 3|$$

If you are using the column notation for the kets you make a row vector under the adjoint so

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^\dagger = (v_1 \; v_2 \; v_3)$$

This all agrees nicely with the linear algebra conventions. Note that the adjoint is the "complex transpose" in that context. This operation is sometimes also called the "Hermitian conjugate".

The scalar product or *inner product* is written as $\langle \cdot \mid \cdot \rangle$. This has many other interpretations as well. The most important interpretation in quantum mechanics is (fanfare!)

The inner product is the probability amplitude.

So this is the beastie which gives the predictions (such as they are). If you have a state $\mid \psi\rangle$ and what to find out whether the spin is up in the z direction then you calculate

$$|\langle +z \mid \psi\rangle|^2 = \langle +z \mid \psi\rangle\langle +z \mid \psi\rangle^*$$

and that is the *probability*.

The component, or representation, of a ket vector $\mid a\rangle$ in the basis $\mid i\rangle$ is

$$(\mid i\rangle)^\dagger \mid a\rangle = \langle i \mid a\rangle$$

So

$$\mid a\rangle = \sum_i a_i \mid i\rangle = \sum_i \mid i\rangle\langle i \mid a\rangle$$

In quantum mechanics, you can write a wavefunction $\psi(x)$ as the wavefunction in the x representation, i.e. $\langle x \mid \psi\rangle$.

You can now write 1 in a new way

$$\sum_i \mid i\rangle\langle i \mid = 1!$$

This states that the basis $\mid i\rangle$ is complete.

Operators, often written with hats, $\hat{}$ (in polite company), take a ket and produce another

$$\hat{Q} \mid a\rangle = \mid b\rangle$$

You can express this as a matrix operation by working in a basis like $\mid i\rangle$. The operator is entirely determined by how it acts on every basis vector

$$\hat{Q} \mid i\rangle = \sum_j Q_{ij} \mid j\rangle$$