$\frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$. Determine the moment of the force about the a) x axis, and b) a line oriented by the unit vector $\mathbf{i}_L = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$.

6.3 Vector Differentiation

6.3.1 ORDINARY DIFFERENTIATION

We study vector functions of one or more scalar variables. In this section we examine differentiation of vector functions of one variable. The derivative of the vector $\mathbf{u}(t)$ with respect to t is defined, as usual, by

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t}, \tag{6.3.1}$$

where

$$\mathbf{u}(t + \Delta t) - \mathbf{u}(t) = \Delta \mathbf{u}. \tag{6.3.2}$$

This is illustrated in Fig. 6.11. Note that the direction of $\Delta\mathbf{u}$ is, in general, unrelated to the direction of $\mathbf{u}(t)$.

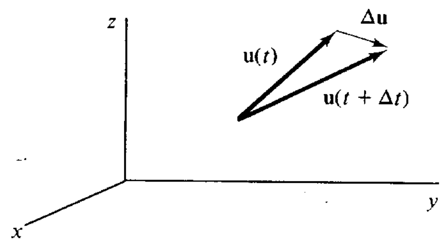

FIGURE 6.11. Vectors used in the definition of the derivative $d\mathbf{u}/dt$.

From this definition it follows that the sums and products involving vector quantities can be differentiated as in ordinary calculus; that is,

$$\frac{d}{dt}(\phi\mathbf{u}) = \phi\frac{d\mathbf{u}}{dt} + \mathbf{u}\frac{d\phi}{dt}$$

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\frac{d\mathbf{u}}{dt} \tag{6.3.3}$$

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times v.$$

If we express the vector $\mathbf{u}(t)$ in rectangular coordinates, as

$$\mathbf{u}(t) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}, \tag{6.3.4}$$

if can be differentiated term by term to yield

$$\frac{d\mathbf{u}}{dt} = \frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k} \tag{6.3.5}$$

provided that the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are independent of t. If t represents time, such a reference frame is referred to as an *inertial reference frame*.

We shall illustrate differentiation by considering the motion of a particle in a noninertial reference frame. Let us calculate the velocity and acceleration of such a particle. The particle occupies the position (x, y, z) measured in the noninertial xyz reference frame which is rotating with an angular velocity $\boldsymbol{\omega}$, as shown in Fig. 6.12. The xyz reference frame is located by the vector $\mathbf{s}$ relative to the inertial XYZ reference frame.* The velocity $\mathbf{V}$ referred to the XYZ frame is

$$\mathbf{V} = \frac{d}{dt}(\mathbf{s} + \mathbf{r}) = \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt}. \tag{6.3.6}$$

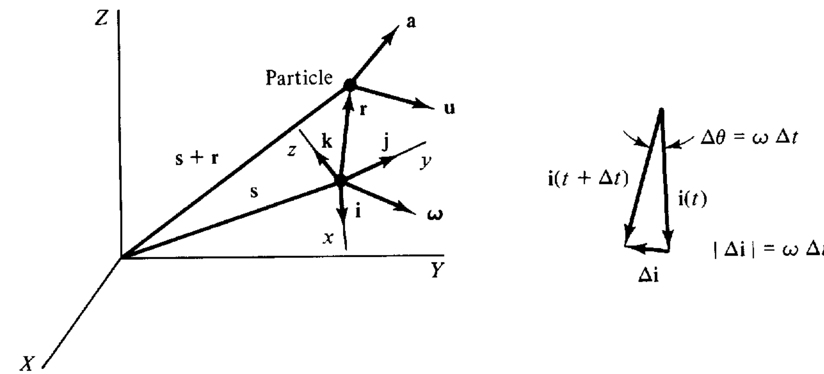

FIGURE 6.12. Motion referred to a noninertial reference frame.

The quantity $d\mathbf{s}/dt$ is the velocity of the xyz reference frame and is denoted $\mathbf{V}_{\text{ref}}$. The vector $d\mathbf{r}/dt$ is, using $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}. \tag{6.3.7}$$

To determine an expression for the time derivatives of the unit vectors, which are due to the angular velocity $\boldsymbol{\omega}$ of the xyz frame, consider the unit vector $\mathbf{i}$ to rotate through a small angle during the time Δt, illustrated in Fig. 6.12.

*This reference frame is attached to the ground in the case of a projectile or a rotating device; it is attached to the sun when describing the motion of satellites.

Using the definition of a derivative, there results

$$\frac{d\mathbf{i}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{i}(t + \Delta t) - \mathbf{i}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta \mathbf{i}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\omega \Delta t \left(\dfrac{\boldsymbol{\omega} \times \mathbf{i}}{\omega} \right)}{\Delta t} = \boldsymbol{\omega} \times \mathbf{i}, \tag{6.3.8}$$

where the quantity $\boldsymbol{\omega} \times \mathbf{i}/\omega$ is a unit vector perpendicular to $\mathbf{i}$ in the direction of $\Delta\mathbf{i}$. Similarly,

$$\frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j}, \qquad \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \times \mathbf{k}. \tag{6.3.9}$$

Substituting these and Eq. 6.3.7 into Eq. 6.3.6, we have

$$\mathbf{V} = \mathbf{V}_{\text{ref}} + \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} + x\boldsymbol{\omega} \times \mathbf{i} + y\boldsymbol{\omega} \times \mathbf{j} + z\boldsymbol{\omega} \times \mathbf{k}. \tag{6.3.10}$$

The velocity $\mathbf{v}$ of the particle relative to the xyz frame is

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \tag{6.3.11}$$

Hence, we can write the expression for the absolute velocity as

$$\mathbf{V} = \mathbf{V}_{\text{ref}} + \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}. \tag{6.3.12}$$

The absolute acceleration $\mathbf{A}$ is obtained by differentiating $\mathbf{V}$ with respect to time to obtain

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}_{\text{ref}}}{dt} + \frac{d\mathbf{v}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}. \tag{6.3.13}$$

In this equation

$$\frac{d\mathbf{V}_{\text{ref}}}{dt} = \mathbf{A}_{\text{ref}} \tag{6.3.14}$$

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} + v_x\frac{d\mathbf{i}}{dt} + v_y\frac{d\mathbf{j}}{dt} + v_z\frac{d\mathbf{k}}{dt}$$

$$= \mathbf{a} + \boldsymbol{\omega} \times \mathbf{v} \tag{6.3.15}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}, \tag{6.3.16}$$

where $\mathbf{a}$ is the acceleration of the particle observed in the xyz frame. The absolute acceleration is then

$$\mathbf{A} = \mathbf{A}_{\text{ref}} + \mathbf{a} + \boldsymbol{\omega} \times \mathbf{v} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}). \tag{6.3.17}$$

This is reorganized in the form

$$\mathbf{A} = \mathbf{A}_{\text{ref}} + \mathbf{a} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}. \tag{6.3.18}$$

The quantity $2\boldsymbol{\omega} \times \mathbf{v}$ is often referred to as the *Coriolis acceleration*, and $d\boldsymbol{\omega}/dt$ is the angular acceleration of the xyz frame. For a rigid body $\mathbf{a}$ and $\mathbf{v}$ are zero.

Example 6.3.1: Using the definition of a derivative, show that

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \frac{d\mathbf{u}}{dt}.$$

SOLUTION: The definition of a derivative allows us to write

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t + \Delta t) \cdot \mathbf{v}(t + \Delta t) - \mathbf{u}(t) \cdot \mathbf{v}(t)}{\Delta t}.$$

But we know that (see Fig. 6.11)

$$\mathbf{u}(t + \Delta t) - \mathbf{u}(t) = \Delta\mathbf{u}$$

$$\mathbf{v}(t + \Delta t) - \mathbf{v}(t) = \Delta\mathbf{v}.$$

Substituting for $\mathbf{u}(t + \Delta t)$ and $\mathbf{v}(t + \Delta t)$, there results

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \lim_{\Delta t \to 0} \frac{[\Delta\mathbf{u} + \mathbf{u}(t)] \cdot [\Delta\mathbf{v} + \mathbf{v}(t)] - \mathbf{u}(t) \cdot \mathbf{v}(t)}{\Delta t}.$$

This product is expanded to yield

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{u} \cdot \Delta\mathbf{v} + \mathbf{u} \cdot \Delta\mathbf{v} + \mathbf{v} \cdot \Delta\mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v}}{\Delta t}.$$

In the limit as $\Delta t \to 0$, both $\Delta\mathbf{u} \to \mathbf{0}$ and $\Delta\mathbf{v} \to \mathbf{0}$. Hence,

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{u} \cdot \Delta\mathbf{v}}{\Delta t} \to 0.$$

We are left with

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \lim_{\Delta t \to 0} \left(\mathbf{u} \cdot \frac{\Delta\mathbf{v}}{\Delta t} + \mathbf{v} \cdot \frac{\Delta\mathbf{u}}{\Delta t} \right)$$

$$= \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \frac{d\mathbf{u}}{dt}$$

and the given relationship is proved. ■

Example 6.3.2: The position of a particle is given by $\mathbf{r} = t^2\mathbf{i} + 2\mathbf{j} + 5(t - 1)\mathbf{k}$ meters, measured in a reference frame that has no translational velocity but that has an angular velocity of 20 rad/s about the z axis. Determine the absolute velocity at $t = 2$ s.

SOLUTION: Given that $\mathbf{V}_{\text{ref}} = \mathbf{0}$ the absolute velocity is

$$\mathbf{V} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity, as viewed from the rotating reference frame, is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}[t^2\mathbf{i} + 2\mathbf{j} + 5(t - 1)\mathbf{k}]$$
$$= 2t\mathbf{i} + 5\mathbf{k}.$$

The contribution due to the angular velocity is

$$\boldsymbol{\omega} \times \mathbf{r} = 20\mathbf{k} \times [t^2\mathbf{i} + 2\mathbf{j} + 5(t - 1)\mathbf{k}]$$
$$= 20t^2\mathbf{j} - 40\mathbf{i}.$$

Thus, the absolute velocity is

$$\mathbf{V} = 2t\mathbf{i} + 5\mathbf{k} + 20t^2\mathbf{j} - 40\mathbf{i}$$
$$= (2t - 40)\mathbf{i} + 20t^2\mathbf{j} + 5\mathbf{k}.$$

At $t = 2$ s this becomes

$$\mathbf{V} = -36\mathbf{i} + 80\mathbf{j} + 5\mathbf{k} \quad \text{m/s.} \quad \blacksquare$$

Example 6.3.3: A person is walking toward the center of a merry-go-round along a radial line at a constant rate of 6 m/s. The angular velocity of the merry-go-round is 1.2 rad/s. Calculate the absolute acceleration when the person reaches a position 3 m from the axis of rotation.

SOLUTION: The acceleration $\mathbf{A}_{\text{ref}}$ is assumed to be zero, as is the angular acceleration $d\boldsymbol{\omega}/dt$ of the merry-go-round. Also, the acceleration $\mathbf{a}$ of the person relative to the merry-go-round is zero. Thus, the absolute acceleration is

$$\mathbf{A} = 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Attach the xyz reference frame to the merry-go-round with the $\mathbf{z}$ axis vertical and the person walking along the x axis toward the origin. Then

$$\boldsymbol{\omega} = 1.2\mathbf{k}, \qquad \mathbf{r} = 3\mathbf{i}, \qquad \mathbf{v} = -6\mathbf{i}.$$

The absolute acceleration is then

$$\mathbf{A} = 2[1.2\mathbf{k} \times (-6\mathbf{i})] + 1.2\mathbf{k} \times (1.2\mathbf{k} \times 3\mathbf{i})$$
$$= -4.32\mathbf{i} - 14.4\mathbf{j} \quad \text{m/s}^2.$$

Note the y component of acceleration that is normal to the direction of motion, which makes the person sense a tugging in that direction. $\blacksquare$

6.3.2 PARTIAL DIFFERENTIATION

Many phenomena require that a quantity be defined at all points of some region of interest. The quantity may also vary with time. Such quantities are often referred to as *field quantities:* electric fields, magnetic fields, velocity fields, and pressure fields are examples. Partial derivatives are necessary when describing fields. Consider a vector function $\mathbf{u}(x, y, z, t)$.

The partial derivative of $\mathbf{u}$ with respect to x is defined to be

$$\frac{\partial \mathbf{u}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{u}(x + \Delta x, y, z, t) - \mathbf{u}(x, y, z, t)}{\Delta x}. \tag{6.3.19}$$

In terms of the components we have

$$\frac{\partial \mathbf{u}}{\partial x} = \frac{\partial u_x}{\partial x}\mathbf{i} + \frac{\partial u_y}{\partial x}\mathbf{j} + \frac{\partial u_z}{\partial x}\mathbf{k} \tag{6.3.20}$$

where each component could be a function of x, y, z, and t.

The incremental quantity $\Delta\mathbf{u}$ between the two points (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$ at the same instant in time is

$$\Delta\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x}\Delta x + \frac{\partial \mathbf{u}}{\partial y}\Delta y + \frac{\partial \mathbf{u}}{\partial z}\Delta z. \tag{6.3.21}$$

At a fixed point in space $\Delta\mathbf{u}$ is given by

$$\Delta\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t}\Delta t. \tag{6.3.22}$$

If we are interested in the acceleration of a particular particle in a region fully occupied by particles, a *continuum,* we write the incremental velocity $\Delta\mathbf{v}$ between two points, shown in Fig. 6.13, as

$$\Delta\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x}\Delta x + \frac{\partial \mathbf{v}}{\partial y}\Delta y + \frac{\partial \mathbf{v}}{\partial z}\Delta z + \frac{\partial \mathbf{v}}{\partial t}\Delta t, \tag{6.3.23}$$

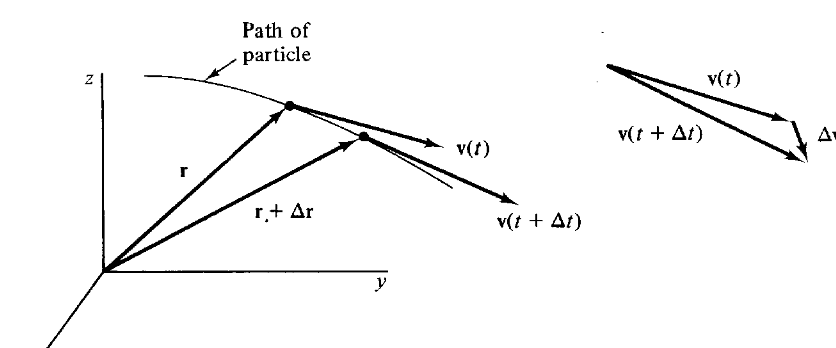

FIGURE 6.13. Motion of a particle.