Topics in Mathematical Physics (PHYS 320): PS 1 — Spring 2017

Welcome to the first problem set, primarily on ODE's!

- Please submit your solutions in class on Thursday February 16.
- Please use your notes, Mathematica, Wolfram Alpha, and Boas, but no other resources. Please *cite any references* (source, page number and formula number) and include printouts of your Mathematica notebook(s), as appropriate.
- Your solutions must be entirely your own work.
- Unlike the daily solutions I ask that you do not work with others, but please ask me questions if a question is unclear, if there is confusion, or if you are unsure how to proceed.
- **Please check your solutions.**

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(1) A tube with a hole is in our classroom. In this problem we get to play and to determine the equation of motion and solution for the water level inside the tube.
  - (a) Model the system and find the differential equation that will determine the height of water in the tube as a function of time.
  - (b) Determine the quantities you need to measure and measure them.
  - (c) Find the general solution to the differential equation.
  - (d) Find the specific solution for the initial condition that the water starts at the top mark on the container at $t = 0$.
  - (e) Measure this time and verify your result.

(2) Use Mathematica to explore the solutions and solution space of

$$u'(x) = -xu + \frac{8x}{u}$$

using the slope field techniques. Include solutions on your slope field. Discuss what you find.

(3) Show that

$$3u^2u' + x^2 = 0$$

is exact and find a solution.

(4) Complex numbers
  - (a) If

$$z = e^{3\ln 2 - i\pi}$$

  then what are x and y in $z = x + iy$ and r and θ in the polar form $re^{i\theta}$?
  - (b) If $w = \sqrt[3]{z}$, find the value(s) of w and sketch the(se) point(s) in the complex plane.

(5) Consider the differential equation for $u(x)$

$$u\frac{du}{dx} + u^2 - 8x + 2 = 0.$$

  - (a) Describe this equation.
  - (b) Solve by any method you wish.
  - (c) Find the solution with the condition $u(-1) = 2.516$.
  - (d) Comment on the nature of the initial value problem for any condition $u(x_o) = u_o$.

(6) Consider the initial value problem

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = -5\cos(3.16t) \text{ with } y(0) = 0.1 \text{ and } \dot{y}(0) = 4.$$

(a) Find the solution by any method.
(b) Plot the solution at 'early times' $0 < t < 6$ and at late times.
(c) Discuss the differential equation and solution.

(7) It is raining as I write this and so I wonder about those raindrops - what is their acceleration as they fall through a cloud?

(a) The raindrops accumulate mass as they fall. Use $F = dp/dt$ or $F\Delta t = \Delta p$ to derive the relation

$$v\frac{dm}{dt} + m\frac{dv}{dt} = mg.$$

(b) Show that the rate at which raindrops accumulate mass is given by

$$\frac{dm}{dt} = a\rho^{1/3}vm^{2/3}$$

where ρ is the cloud's density and a is a numerical coefficient that, upon successfully deriving this relation, you will find.

(c) There are lots of 'moving parts' in the resulting differential equation - t, m, v, x - ick! But there's a change of variables that makes the solution easier. Note that if we think of the acceleration

$$\frac{dv}{dt} \text{ as } \frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt}$$

we have

$$\left(v + m\frac{dv}{dm}\right)\frac{dm}{dt} = mg.$$

Ah, so we should use m as the independent variable! Substituting the result for dm/dt and letting $u = v^2$ (where $u = u(m)$) show that you obtain the first order equation

$$\frac{du}{dm} + \frac{2}{m}u = b\frac{g}{m^{2/3}}$$

where b is a coefficient that you will find during the derivation.

(d) Find the solution to this differential equation using integration factors.
(e) EXTRA: If you'd like (and for extra points) determine the acceleration of the raindrop as it falls through the cloud, assuming that the initial mass vanishes when the raindrop starts from rest.