Topics in Mathematical Physics (PHYS 320): PS 4 Spring 2017

Welcome to the final problem set!

- Please submit your solutions by 5 PM, Friday May 12.
- Please use your notes, Mathematica, Wolfram Alpha, and Boas, but no other resources. Include printouts of your work with these programs.
- Please *cite any references* (source, page number and formula number, as appropriate).
- You may not consult any other resources such as the internet.
- Your solutions must be entirely your own work.
- Please check your results.

---

(1) (10 pts.) Solve the initial value problem

$$u'' + 4u' + 4u = 0 \text{ with } u(0) = 0 \text{ and } u'(0) = 2.$$

(2) (20 pts.) Consider the ode

$$u' \sin x = -u \ln u$$

(a) Describe this equation.
(b) Solve this equation by any method you would like.
(c) Solve the initial value problem for the initial condition $u(\pi/3) = e$.
(d) Use mathematica to plot the slope field and your solution. Please include a printout of your solution.
(e) Is the initial value problem well-posed? If so, name the theorem that guarantees this. If not, give two other solutions that are of a different form. Include a printout of these solutions on the slope field.

(3) (15 pts.) Playing with tops on a rainy rain you encounter this pair of coupled odes

$$\frac{d\omega_x}{dt} = -a\omega_y$$
$$\frac{d\omega_y}{dt} = a\omega_x$$

where the ω's are angular velocities,

$$a = \frac{I_z - I}{I}\omega_z,$$

(which I include only for general interest), and the I's are moments of inertia. With initial conditions $\omega_x(0) = \omega$ and $\omega_y(0) = 0$, solve these coupled equations using Laplace transforms.

(4) (30 pts.) Consider the differential equation

$$u'' + \left(\frac{\lambda}{x} - \frac{1}{4} - \frac{\ell(\ell+1)}{x^2}\right)u = 0,$$

where ℓ is a non-vanishing integer and x is a positive real number. We'll solve the eigenvalue problem for λ.

(a) The function u satisfies $u \to 0$ as $x \to \infty$. Find the solution in asymptopia.
(b) The function u is finite at the origin. Solve the ode in the neighborhood of $x = 0$.
(c) Use the results of the previous parts to change dependent variables from $u(x)$ to $v(x)$. Verify that you then obtain the ode

$$xv'' + (2\ell + 2 - x)v' + (\lambda - \ell - 1)v = 0$$

(d) Is this linear operator self-adjoint? If not, what function would make it self-adjoint?
(e) Solve this ode.
(f) Show that if the solution $u(x)$ satisfies the BC's then λ is an integer.
(g) Name this special function! How is λ related to the usual integer label?

(5) (35 pts.) To find the electric (or scalar) potential inside a spherical surface of radius R you can solve the Laplace equation

$$\nabla^2 V = 0.$$

Let's assume that surface is maintained at a constant, non-vanishing potential V_o on $\pi > \theta > \pi/2$ and 0 V on $0 < \theta < \pi/2$.

(a) In a region with no charges, derive the above partial differential equation using Gauss's law,

$$\nabla \cdot \vec{E} = \rho/\epsilon_o,$$

and the definition of the electric potential $\vec{E} = -\nabla V$.
(b) Write down the partial differential equation in terms of spherical coordinates.
(c) What can you say about the ϕ part of the equation? There is a nice simplification here. Please do not go on until you have found it. Please ask if you are puzzled.
(d) Separate variables and solve the resulting odes.
(e) Write down a general solution for $V(r, \theta)$.
(f) Determine the specific solution for the boundary conditions given above.

(6) (Optional) Compose a poem on one or more mathematical methods of 320. For instance you might write a soliloquy for your favorite Bessel function.