QPS 3 SOLUTIONS

(1.) THE IVP $u'' + 4u' + 8u = 0$, $u(0) = 2$, $u'(0) = 0$

TAKING THE LAPLACE TRANSFORM

$$s^2 \mathcal{U} - s\,\overset{=2}{\cancel{u(0)}} + 4\left[s\mathcal{U} - \overset{=2}{\cancel{u(0)}} \right] + 8\,\mathcal{U} = 0$$

$$\Rightarrow \mathcal{U}(s) = \frac{2s+8}{s^2+4s+8} = \frac{2(s+2)}{(s+2)^2+4} + \frac{4}{(s+2)^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}[\mathcal{U}] = u(t) = 2e^{-2t}\left(\cos 2t + \sin 2t\right)$$

(2.) THE IVP $u'' + 2u' + 10u = -6e^{-x}\sin(3x)$, $u(0) = 2$, $u'(0) = 1$

TAKING THE LAPLACE TRANSFORM

$$s^2 \mathcal{U} - s u(0) - u'(0) + 2\left[s\mathcal{U} - u(0) \right] + 10\,\mathcal{U} = -6 \frac{3}{(s+1)^2+3^2}$$

$$\Rightarrow s^2\mathcal{U} - s2 - 1 + 2s\mathcal{U} - 4 + 10\,\mathcal{U} = \frac{-18}{(s+1)^2+3^2}$$

[BOAS L13]

OR $$\left(s^2+2s+10\right)\mathcal{U} - (2s+5) = \frac{-18}{(s+1)^2+3^2}$$

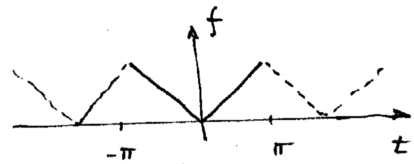
$$\therefore \mathcal{U}(s) = \frac{-18}{\left[(s+1)^2+3^2\right](s^2+2s+10)} + \frac{2s+5}{(s^2+2s+10)} = \frac{-18}{\left[(s+1)^2+9\right]^2} + \frac{2s+5}{(s+1)^2+9}$$

USING MATHEMATICA TO TAKE THE INVERSE I FIND

$$\mathcal{L}^{-1}[\mathcal{U}] = u(x) = \frac{1}{6}e^{-x}\left[6(2+x)\cos 3x + 4\sin 3x\right]$$

(SEE PRINTOUT)

(3.) PERIODIC EXTENSION



$f(t)$ IS AN EVEN FUNCTION SO $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -t\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\,dt = \frac{2}{\pi}\int_0^{\pi} t\,dt$$

$$= \frac{2}{\pi}\,\frac{\pi^2}{2} = \pi$$

FOR $n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -t\cos(nt)\,dt + \frac{1}{\pi}\int_0^{\pi} t\cos(nt)\,dt$$

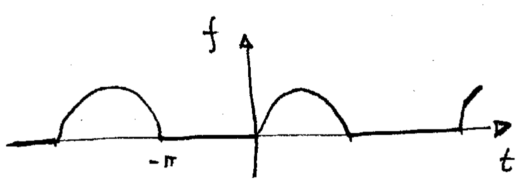
$$= \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \left(\frac{2}{\pi}\right)\left(-\frac{1}{n^2} + \frac{\cos(n\pi)}{n^2} + \cancel{\frac{\pi\sin(n\pi)}{n}}^{\,0}\right)$$ [FROM MATHEMATICA]

$$= \left(\frac{2}{\pi}\right)\left(-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right) = -\frac{4}{\pi}\,\frac{1}{n^2} \quad \text{FOR } \underline{\text{ODD}}\ n.$$

$$\therefore\ f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x$$

(4.) THE PERIODIC EXTENSION OF $f(t) = \begin{cases} \sin t & 0 \leq t \leq \pi \\ 0 & \pi \leq t \leq 2\pi \end{cases}$



$$a_0 = \frac{1}{\pi}\int_0^{\pi}\sin t\,dt = \frac{1}{\pi}(-\cos t)\Big]_0^{\pi} = \frac{2}{\pi}$$

FOR $n \neq 1$,

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin t \cos nt \, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin(t+nt) + \sin(t-nt)] \, dt$$

$$= -\frac{1}{2\pi}\left[\frac{\cos(n+1)t}{n+1} - \frac{\cos(n-1)t}{n-1} \right]_0^{\pi} = \frac{-1}{2\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} \right] + \frac{1}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi(n^2-1)}\left((-1)^{n+1} - 1\right) \qquad \left[= \begin{cases} 0 & \text{IF } n \text{ IS ODD} \\ \dfrac{-2}{\pi(n^2-1)} & \text{IF } n \text{ IS EVEN} \end{cases} \right]$$

WHILE

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin t \sin nt \, dt = \frac{1}{2\pi}\int_0^{\pi} [-\cos(n+1)t + \cos(n-1)t] \, dt$$

$$= \frac{1}{2\pi}\left[\frac{-\sin(n+1)t}{n+1} + \frac{\sin(n-1)t}{n-1} \right]_0^{\pi} = 0$$

BUT THESE EXPRESSIONS ARE NOT DEFINED FOR $n=1$ SO WORKING THESE OUT

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin t \cos t \, dt = \frac{1}{\pi} \frac{\sin^2 t}{2}\Big]_0^{\pi} = 0 \qquad \text{WHILE}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t \, dt = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

$$\therefore \; f(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2-1}$$

OR

$$= \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{1}{\pi}\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{(2)\cos nt}{n^2-1}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2\cos 2t}{3\pi} - \frac{2\cos 4t}{15\pi} - \cdots$$

(5.) $\Sigma F = ma$ GIVES

$$-b\dot{y} - ky + f(t) = m\ddot{y} \quad \Rightarrow \quad \ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{f(t)}{m}$$

WITH THE NUMBERS GIVEN THEN

$$\ddot{y} + 4\dot{y} + 38y = \frac{2}{2}u_2(t) \qquad \text{AND} \quad y(0) = y'(0) = 0$$

TAKING THE LAPLACE TRANSFORM

$$s^2Y + 4sY + 38Y = \frac{e^{-2s}}{s} \quad \Rightarrow \quad Y = \frac{e^{-2s}}{s(s^2+4s+38)}$$

HENCE,

$$\mathcal{L}^{-1}[Y] = y(t) = u_2(t)\left[\frac{1}{38} - \frac{1}{646}e^{-2(-2+t)}\left(17\cos(\sqrt{34}(t-2)) + \sqrt{34}\sin(\sqrt{34}(t-2))\right)\right]$$

SEE MATHEMATICA NOTES

In[28]:= (* For problem 2 *)

In[29]:= InverseLaplaceTransform[
(2 * s + 5) / (s ^ 2 + 2 * s + 10) - 18 / (((s + 1) ^ 2 + 9) * (s ^ 2 + 2 * s + 10)), s, x]

Out[29]= $\frac{1}{6} e^{(-1-3 i) x} ((6+2 i) + 3 x + e^{6 i x} ((6-2 i) + 3 x))$

In[30]:= ComplexExpand[%]

Out[30]= $e^{-x} \text{Cos}[3 x] + \frac{1}{2} e^{-x} x \text{Cos}[3 x] + e^{-x} \text{Cos}[3 x] \text{Cos}[6 x] + \frac{1}{2} e^{-x} x \text{Cos}[3 x] \text{Cos}[6 x] + \frac{1}{3} e^{-x} \text{Sin}[3 x] - \frac{1}{3} e^{-x} \text{Cos}[6 x] \text{Sin}[3 x] + \frac{1}{3} e^{-x} \text{Cos}[3 x] \text{Sin}[6 x] + e^{-x} \text{Sin}[3 x] \text{Sin}[6 x] + \frac{1}{2} e^{-x} x \text{Sin}[3 x] \text{Sin}[6 x] + i \left(\frac{1}{3} e^{-x} \text{Cos}[3 x] - \frac{1}{3} e^{-x} \text{Cos}[3 x] \text{Cos}[6 x] - e^{-x} \text{Sin}[3 x] - \frac{1}{2} e^{-x} x \text{Sin}[3 x] - e^{-x} \text{Cos}[6 x] \text{Sin}[3 x] - \frac{1}{2} e^{-x} x \text{Cos}[6 x] \text{Sin}[3 x] + e^{-x} \text{Cos}[3 x] \text{Sin}[6 x] + \frac{1}{2} e^{-x} x \text{Cos}[3 x] \text{Sin}[6 x] - \frac{1}{3} e^{-x} \text{Sin}[3 x] \text{Sin}[6 x]\right)$

In[31]:= $\text{Simplify}\left[e^{-x} \text{Cos}[3 x] + \frac{1}{2} e^{-x} x \text{Cos}[3 x] + e^{-x} \text{Cos}[3 x] \text{Cos}[6 x] + \frac{1}{2} e^{-x} x \text{Cos}[3 x] \text{Cos}[6 x] + \frac{1}{3} e^{-x} \text{Sin}[3 x] - \frac{1}{3} e^{-x} \text{Cos}[6 x] \text{Sin}[3 x] + \frac{1}{3} e^{-x} \text{Cos}[3 x] \text{Sin}[6 x] + e^{-x} \text{Sin}[3 x] \text{Sin}[6 x] + \frac{1}{2} e^{-x} x \text{Sin}[3 x] \text{Sin}[6 x]\right]$

Out[31]= $\frac{1}{6} e^{-x} (6 (2 + x) \text{Cos}[3 x] + 4 \text{Sin}[3 x])$

In[32]:= (* Checking this solution *)

In[33]:= DSolve[{u''[x] + 2 * u'[x] + 10 * u[x] == -6 * Exp[-x] * Sin[3 * x], u[0] == 2, u'[0] == 1}, u, x]

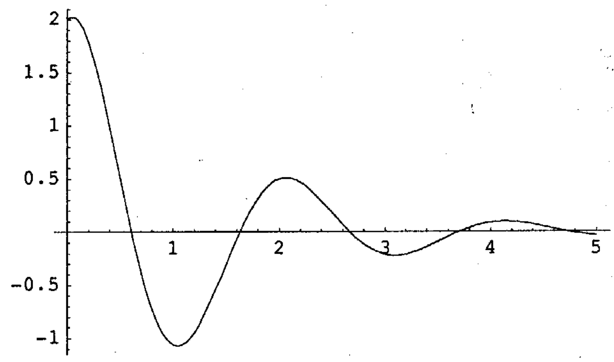
Out[33]= $\{\{u \to \text{Function}[\{x\}, \frac{1}{6} e^{-x} (12 \text{Cos}[3 x] + 6 x \text{Cos}[3 x] + 5 \text{Sin}[3 x] + \text{Cos}[6 x] \text{Sin}[3 x] - \text{Cos}[3 x] \text{Sin}[6 x])]\}\}$

In[34]:= $\text{Simplify}\left[\frac{1}{6} e^{-x} (12 \text{Cos}[3 x] + 6 x \text{Cos}[3 x] + 5 \text{Sin}[3 x] + \text{Cos}[6 x] \text{Sin}[3 x] - \text{Cos}[3 x] \text{Sin}[6 x])\right]$

Out[34]= $\frac{1}{6} e^{-x} (6 (2 + x) \text{Cos}[3 x] + 4 \text{Sin}[3 x])$

In[35]:= (* Hooray! It looks right. Here's what the function looks like *)

In[36]:= Plot$\left[\frac{1}{6} e^{-x} (6 (2 + x) \text{Cos}[3 x] + 4 \text{Sin}[3 x]), \{x, 0, 5\}\right]$



Out[36]= - Graphics -

In[41]:= (* For problem 3 *)

In[42]:= Integrate[t * Cos[n * t], {t, 0, Pi}]

Out[42]= $-\frac{1}{n^2} + \frac{\text{Cos}[n \pi]}{n^2} + \frac{\pi \text{Sin}[n \pi]}{n}$

In[43]:= (* For problem 5 *)

In[44]:= InverseLaplaceTransform[Exp[-2 * s] / (s (s^2 + 4 * s + 38)), s, t]

Out[44]= $\left(\frac{1}{38} - \frac{1}{646} e^{-2 (-2+t)} (17 \text{Cos}[\sqrt{34} (-2 + t)] + \sqrt{34} \text{Sin}[\sqrt{34} (-2 + t)])\right) \text{UnitStep}[-2 + t]$