PHYS 320 : QPS 2 - ODEs

(1) FOR $3u^2\,du + x^2\,dx = N\,du + M\,dx = 0$ WE HAVE AN EXACT EQU'N IF $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial u}$. CHECKING FOR $N = 3u^2$ AND $M = x^2$ WE HAVE $0=0$ - THE EQU'N IS EXACT. NOW SOLVING

$$N = \frac{\partial \Phi}{\partial u} = 3u^2 \Rightarrow \Phi = u^3 + f(x).$$ NOW FOR M

$$M = \frac{\partial \Phi}{\partial x} = f' = x^2 \Rightarrow f = \int x^2\,dx = \frac{x^3}{3} \quad \text{SO} \quad \Phi = u^3 + \frac{x^3}{3} = C.$$

$$\Rightarrow u = \sqrt[3]{C - \frac{x^3}{3}}$$

NOW CHECKING THIS WITH THE ORIGINAL ODE $3u^2 u' + x^2 = 0$. IMPLICITLY DIFF Φ: $3u^2\,du + x^2\,dx = 0 \Rightarrow$ THIS SOL'N WORKS!

(2) (a) 1^{ST} ORDER, NON-LINEAR DIFF. EQU'N. IT IS SEPERABLE

(b) & (c) FROM THE SLOPE FIELDS ON THE NEXT PAGE IT IS CLEAR THAT $\pm 2\sqrt{2}$ ARE ATTRACTORS. LET'S SEE IF THIS COMPLICATES THE IVPs. SEPARATING I HAVE

$$\int \frac{y\,dy}{y^2-8} = -\int x\,dx \quad \text{OR} \quad \int \frac{\frac{1}{2}\,dy^2}{y^2-8} = -\frac{x^2}{2} + c'$$

$$\Rightarrow \ln(y^2-8) = -x^2 + c \Rightarrow y^2 - 8 = ke^{-x^2}$$

$$\therefore y(x) = \pm\sqrt{8 + ke^{-x^2}}$$

FOR $y(1) = 3$ THEN $3 = \sqrt{8 + ke^{-1}} \Rightarrow k = e$ SO $y(x) = \sqrt{8 + e^{-x^2+1}}$

THIS APPROACHES $y = 2\sqrt{2}$ BUT ONLY GETS THERE AT $x \to \infty$.

FOR $y(3) = 2\sqrt{2}$, $k = 0$ GIVES THE ONLY SOL'N $y(x) = 2\sqrt{2}$.

(d). THESE IVPs ARE WELL-POSED (ALTHOUGH IT MAY NOT LOOK IT FROM THE SLOPE FIELDS!)

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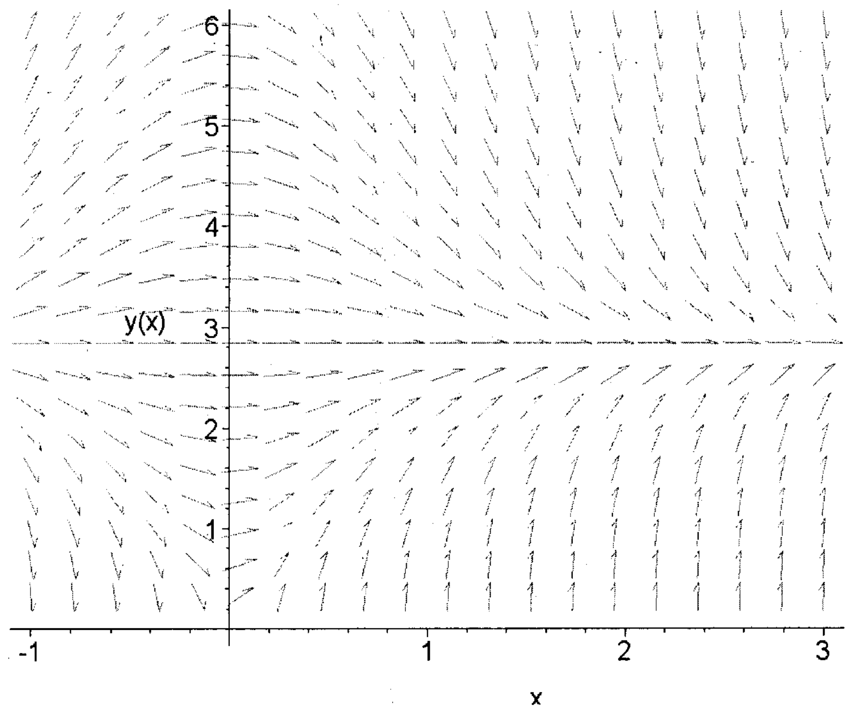
> with(DEtools):
> eqn:=diff(y(x),x)+(x*y(x)^2-8*x)/y(x);
```

$$eqn := \left(\frac{d}{dx}\,y(x)\right) + \frac{x\,y(x)^2 - 8\,x}{y(x)}$$

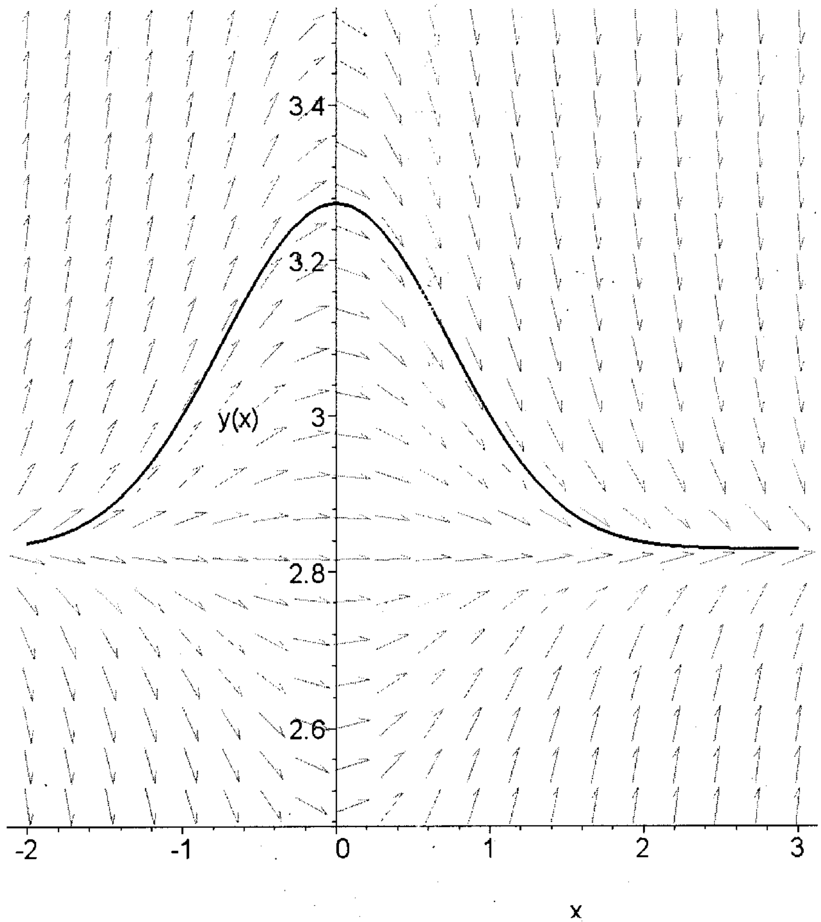
```
> DEplot(eqn,y(x),x=-1..3, y=0..6);
```




```
> dsolve({eqn,y(1)=3},y(x));
```

$$y(x) = \frac{\sqrt{e^{(-1)}\,(8\,e^{(-1)} + e^{(-x^2)})}}{e^{(-1)}}$$

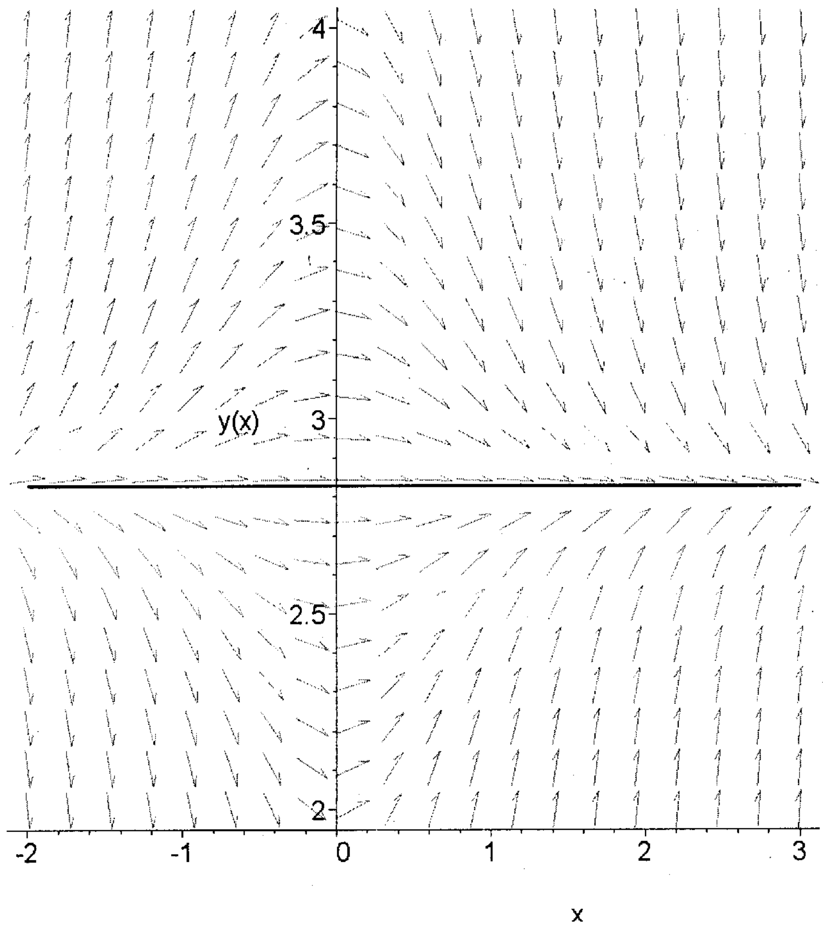
```
> DEplot(eqn,y(x),x=-2..3, y=2.5..3.5,
  {[y(1)=3]},linecolor=black,stepsize=0.1);
```



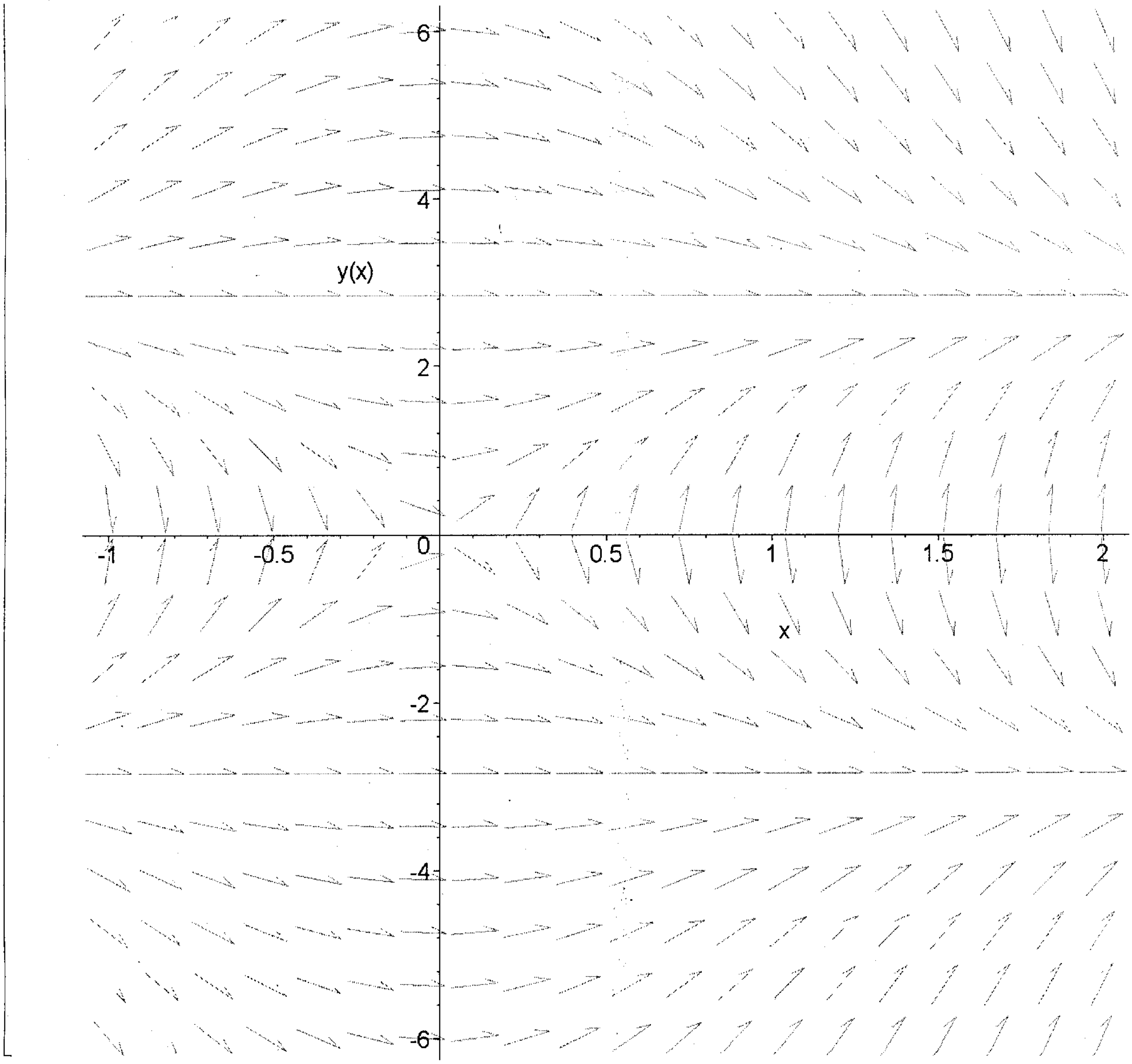

```
> dsolve({eqn,y(3)=sqrt(8)},y(x));
```

$$y(x) = 2\sqrt{2}$$

```
> DEplot(eqn,y(x),x=-2..3, y=2..4,
  {[y(3)=sqrt(8)]},linecolor=black,stepsize=0.1);
```




```
> DEplot(eqn,y(x),x=-1..2, y=-6..6);
```



y(x)
x
6
4
2
-2
-4
-6
-1
-0.5
0
0.5
1
1.5
2

[>

(3) (a) 2^{nd} ORDER, LINEAR, IN-HOMOGENEOUS DIFF. EQU'N WITH CONSTANT COEFFICIENTS.

(b) THE SOL'N IS THE SUM OF THE COMPLEMENTARY AND PARTICULAR SOLUTIONS;

$$u = u_c + u_p$$

STARTING WITH THE HOMOGENEOUS EQU'N

$$u'' + 2u' + 10u = 0$$

I'LL USE THE TRIAL SOL'N $u = e^{mx}$ GIVING

$$\underbrace{(m^2 + 2m + 10)}_{=0} e^{mx} = 0$$

SO THE CHARACTERISTIC EQU'N HAS ROOTS, $m = -1 \pm \frac{\sqrt{4 - 4 \cdot 10}}{\cancel{2}}$ SO u_c MAY BE WRITTEN AS

$$= -1 \pm 3i \Rightarrow u_c = A e^{-x} \cos(3x + \varphi)$$

NOW FOR u_p LET'S TAKE THE $\mathbb{C}$ FORM OF THE ODE

$$\mathcal{U}'' + 2\mathcal{U}' + 10\mathcal{U} = 26 e^{i2x}$$

THE IMAGINARY PART IS THE EQU'N WE WISH TO SOLVE. USING THE TRIAL SOL'N $\mathcal{U} = C e^{i2x}$ WE HAVE

$$(-4C + 4iC + 10C) e^{i2x} = 26 e^{i2x}$$

$$\Rightarrow (6 + 4i) C = 26 \quad \text{OR} \quad (3 + 2i) C = 13$$

SOLVING FOR C GIVES

$$C = \frac{13}{3 + 2i} \frac{(3 - 2i)}{(3 - 2i)} = \frac{\cancel{13}}{\cancel{9+4}} (3 - 2i)$$

HENCE

$$\mathcal{U} = (3 - 2i) e^{i2x} = (3 - 2i)(\cos 2x + i \sin 2x)$$

AND

$$u_p = \mathrm{Im}(\mathcal{U}) = 3 \sin 2x - 2 \cos 2x$$

SO FAR WE HAVE $u = u_c + u_p$ OR

$$u(x) = A e^{-x} \cos(3x + \varphi) + 3 \sin 2x - 2 \cos 2x$$

TO FIND THE SPECIFIC SOL'N WE'LL APPLY ~~BOUNDARY~~ INITIAL CONDITIONS

$$\begin{cases} u(0) = 1 \implies A\cos\varphi - 2 = 1 \implies A\cos\varphi = 3 \\ u'(0) = 0 \implies -A\cos\varphi - 3A\sin\varphi + 6 = 0 \end{cases}$$

USING THE FIRST IN THE SECOND GIVES $-3A\sin\varphi + 3 = 0$ OR $A\sin\varphi = 1$

SO
$$\tan\varphi = \frac{A\sin\varphi}{A\cos\varphi} = \frac{1}{3} \implies \varphi \cong 0.322$$

AND
$$A = \frac{3}{\cos\varphi} \simeq 3.16$$

FINALLY,
$$u = u_c + u_p \cong 3.16\, e^{-x} \cos(3x + 0.322) + 3\sin 2x - 2\cos 2x$$
$$= e^{-x}\left(3\cos 3x - \sin 3x\right) + 3\sin 2x - 2\cos 2x$$

(C) THIS IS AN AMBIGUOUS QUESTION - ONE COULD CHANGE THE 'AMPLITUDE', 26, WITHOUT LIMIT. MORE INTERESTINGLY I COULD TUNE THE DRIVING ANGULAR 'FREQUENCY' TO RESONANCE. FOR THIS SYSTEM

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{10-2} = 2\sqrt{2} \simeq 2.83 \quad [\text{UNITS OF } x]^{-1}$$

($\beta = 1$, $\omega_0^2 = 10$ FROM THE ODE) THIS SYSTEM IS MARGINALLY LIGHTLY DAMPED (MEANING $\beta \ll \omega_0$) SO $\omega_R \neq \omega_0 = 3$. SEE ALSO BOAS PG 426-427.

(4.) (a) 2^{ND} ORDER, LINEAR, HOMOGENEOUS DIFF. EQ'N WITH CONSTANT COEFF.

(b) WITH A TRIAL SOL'N OF THE FORM e^{mx} I FIND

$$m^2 - 4m + 8 = 0 \quad \Rightarrow \quad m = \frac{4}{2} \pm \frac{\sqrt{\cancel{16}^{4} - \cancel{4}\cdot 8}}{\cancel{2}} = 2 \pm i2$$

(NOTE THE NEGATIVE DAMPING TERM) SO

$$u = e^{2x}\left(A\cos 2x + B\sin 2x\right)$$

AS FOR THE INITIAL CONDITIONS WE HAVE

$$\begin{cases} u(0) = 2 \Rightarrow A = 2 \quad \text{AND} \\ u'(0) = 0 \Rightarrow 2A + 2B = 0 \Rightarrow B = -2 \end{cases}$$

HENCE,

$$u(x) = 2e^{2x}\left(\cos 2x - \sin 2x\right)$$

(5.) THIS IS EXACT SINCE

$$(x+u)\,du + (u-x)\,dx = 0 = M\,dx + N\,du$$

$$\Rightarrow M = u - x \quad \text{AND} \quad N = x + u \quad \text{SO} \quad \frac{\partial M}{\partial u} = 1 = \frac{\partial N}{\partial x}$$

SO

$$\frac{\partial \Phi}{\partial u} = x + u \Rightarrow \Phi = xu + \frac{u^2}{2} + f(x). \quad \text{AND}$$

$$\frac{\partial \phi}{\partial x} = u - x \Rightarrow u + f' = u - x \Rightarrow f' = -x \Rightarrow f = -\frac{x^2}{2}$$

$$\therefore \Phi = xu + \frac{u^2}{2} - \frac{x^2}{2} = C \quad \text{OR} \quad u^2 + 2xu - x^2 - C' = 0$$

$$\Rightarrow u = -x \pm \sqrt{x^2 + x^2 + C'} = -x \pm \sqrt{2x^2 + C'}$$

FROM THE SOL'N FOR Φ,

$$d\Phi = dx\, u + du\, dx + u\, du - x\, dx = 0$$

$$\Rightarrow (u+x)\, du + (u-x)\, dx = 0$$ AS ABOVE. THE SOL'N WORKS!

WITH THE (INITIAL) CONDITION, $u(1) = 0$ I HAVE

$$0 = -1 \pm \sqrt{2 + c'} \quad \Rightarrow \quad c' = -1$$ AND WE NEED THE + SIGN

$$\therefore \quad u(x) = -x + \sqrt{2x^2 - 1}$$

(6.) WITH OUR USUAL TRIAL SOL'N OF THE FORM e^{mx} THE ODE GIVES

$$m^2 + 2m + 4 = 0 \quad \Rightarrow \quad m = -1 \pm \sqrt{1-4} = -1 \pm i\sqrt{3}$$

AND SOL'NS ARE OF THE FORM

$$u(x) = e^{-x}\left(A \cos \sqrt{3}x + B \sin \sqrt{3}x \right)$$

THE ICs GIVE

$$\begin{cases} u(0) = 1 & \Rightarrow \quad A = 1 \\ u'(0) = 2 & \Rightarrow \quad -A + \sqrt{3}B = 2 \quad \Rightarrow \quad B = \sqrt{3} \end{cases}$$

$$\therefore \quad u(x) = e^{-x}\left[\cos(\sqrt{3}x) + \sqrt{3} \sin(\sqrt{3}x) \right]$$

(7.) ACHTUNG! THIS IS CRITICALLY DAMPED SINCE $\beta = \omega_0$ (RECALL THAT FOR $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$, CRIT. DAMPED IS $\beta = \omega_0$) SO WE NEED SOL'NS OF THE FORM e^{mx} AND $x e^{mx}$. THE CHARACTERISTIC EQU'N IS $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1$

CHECKING THE SECOND SET OF SOL'NS I HAVE

$$u' = e^{-x} - xe^{-x}$$

$$u'' = -e^{-x} - e^{-x} + xe^{-x} = -2e^{-x} + xe^{-x}$$

THE ODE

$$\Rightarrow \quad \cancel{-2e^{-x}} + \cancel{xe^{-x}} + \cancel{2}e^{-x} - 2x\cancel{e^{-x}} + x\cancel{e^{-x}} = 0 \quad \checkmark$$

HENCE,

$$u(x) = Ae^{-x} + Bxe^{-x}$$

(8.) (a) FOR THE HOMOGENEOUS EQU'N $u'' + u = 0$ WE HAVE $u_1 = \cos x$ AND $u_2 = \sin x$. FOR THE PARTICULAR SOL'N LET'S TRY $u_p = Ae^{-x}$. THE ODE THEN GIVES $Ae^{-x} + Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{2}$

HENCE THE GENERAL SOL'N IS

$$u(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^{-x}$$

(b) THE INITIAL CONDITIONS GIVE

$$\begin{cases} u(0) = 0 & \Rightarrow \quad \frac{1}{2} + C_1 = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{2} \\ u'(0) = 2 & \Rightarrow \quad -\frac{1}{2} + \cancel{\frac{1}{2}\sin 0}^{\,\to 0} + C_2 \cos(0) = 2 \quad \Rightarrow \quad C_2 = \frac{5}{2} \end{cases}$$

So

$$u(x) = \frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{5}{2}\sin x$$

(9.) IN STANDARD FORM THIS EQU'N IS

$$y'' + \frac{2}{x}y' + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)y = 0$$

SO $P_0(x) = \frac{2}{x}$. COMPUTING THE WRONSKIAN DIRECTLY GIVES

$$W(x) = ke^{-\int \frac{2}{x}dx} = ke^{-2\ln x} = \frac{k}{x^2}$$

(10.) (a.) 1^{ST} ORDER, LINEAR HOMOGENEOUS DIFF EQU'N.

(b.) THIS DE IS SEPERABLE

$$u' = u\sin x \quad \Rightarrow \quad \int \frac{du}{u} = \int \sin x \, dx$$

SO $\ln(u) = -\cos x + c$. HENCE $U(x) = Ae^{-\cos x}$

(11.) DESIGNATING x AS THE DISPLACEMENT FROM EQUILIBRIUM, NEWTONIAN MECHANICS GIVES

$$\ddot{x} + w_0^2 x = \frac{0.1}{2}\sin 4t$$

WITH $w_0^2 = \frac{32}{2} = 16 \Rightarrow w_0 = 4\ s^{-1}$. HENCE THIS MASS-SPRING SYSTEM IS BEING DRIVEN AT RESONANCE. THE DOUBLE ROOTS OF THE CHARACTERISTIC EQU'N ($m^2+16=0$) MEAN WE SHOULD TRY SOL'NS OF THE FORM $t\sin(4t)$ AND $t\cos(4t)$. LET'S TRY THE FIRST,

$$\dot{x} = \sin(4t) + 4t\cos(4t)$$

$$\ddot{x} = 4\cos(4t) + 4\cos(4t) - 4t\sin(4t)\cdot 4 = 8\cos(4t) - 16t\sin(4t)$$

THE ODE GIVES (NOW INCLUDING AN AMPLITUDE)

$$A\left[\, 8\cos(4t) - 16t\sin(4t) + \cancel{w_0^2}^{16}\, t\sin(4t)\,\right] = \frac{0.1}{2}\sin 4t$$

LOOKS LIKE THIS WILL ONLY WORK IF I INCLUDE A PHASE (OR SOME OF $t\cos(4t)$). WITH THIS I HAVE

$$A\left[\, 8\cos(4t+\varphi) - 16t\sin(4t+\varphi) + 16t\sin(4t+\varphi)\,\right] = \frac{0.1}{2}\sin(4t)$$

$\Rightarrow \begin{cases} \varphi = \frac{\pi}{2} \\ \text{AND} \\ A = \frac{0.1}{2\cdot 8} \end{cases}$ – MEANING I SHOULD HAVE SOL'NS OF THE FORM $t\cos 4t$. ALSO, THE SYSTEM AT RESONANCE HAS A $\pi/2$ PHASE SHIFT AS WE KNOW FROM PHYS 195

$\therefore \quad X(t) = \underbrace{\frac{0.1}{2.8}}_{\text{THE AMPLITUDE}} t\cos(4t) + x_c(t) \quad \text{WITH} \quad x_c(t) = C\cos(4t) + D\sin(4t)$

THE SPRING BREAKS WHEN THE AMPLITUDE, $\frac{0.1}{2.8} t_c$, IS $\frac{1}{2}$. SO

$$\frac{0.1}{8} t_c = 1 \quad \Rightarrow \quad \underline{\underline{t_c = 80\,s}}$$

(12.) PEEPS $\longrightarrow$ ☺. AS STATED

$$\frac{dV}{dt} = \alpha' A \quad \text{WHERE} \quad \begin{cases} V = \frac{4}{3}\pi r^3 \\ A = 4\pi r^2 \end{cases} \quad \text{FOR THE SPHERE}$$

$$\Rightarrow \quad 3r^2\dot{r} = \alpha' r^2$$

HENCE $\frac{dr}{dt} = \alpha$ AND $r = \alpha t + r_0$ $\quad \left(\alpha = \frac{\alpha'}{3}\right)$

$r_0 = 1\,\text{cm}$. At 6 MONTHS $r = 0.8 = \alpha \cdot 6 + 1 \Rightarrow \alpha = \frac{-.2}{6} = -0.0\bar{3}$

THE RADIUS IS THEN $r = -0.0\bar{3}\, t + 1$, WITH t IN MONTHS.

HENCE, $\frac{1}{4} = -\frac{1}{3} \times 10^{-1} t_{1/4} + 1 \quad \Rightarrow \quad t_{1/4} = 22.5 \simeq 20$ MONTHS.

(b.) MORE QUICKLY SINCE $A_{PEEPS} > A_{SPHERE}$.

(13.) AS STATED

$$\frac{dT}{dt} = C\,(T - T_{env})$$

LET'S SET $t=0$ AT 2:30 PM. WE'LL USE t IN MINUTES. THE ODE IS

$$\frac{dT}{dt} - CT = -C\,T_{env}$$

IT'S SEPERABLE SO

$$\int \frac{dT}{T - T_{env}} = \int c\, dt \quad \Rightarrow \quad \ln(T - T_{env}) = ct + K$$

$$\Rightarrow \quad T = T_0 e^{ct} + T_{env}$$

FROM THE INITIAL CONDITION $T(0) = 190$ I HAVE

$$190 = T_0 + 70 \quad \Rightarrow \quad T_0 = 120$$

At $t = 10$ min

$$150 = 120 e^{c10} + 70 \quad \Rightarrow \quad e^{c10} = \frac{8\cancel{0}}{12\cancel{0}} = \frac{2}{3} \quad \Rightarrow C = \frac{1}{10} \ln \frac{2}{3}$$

$$\simeq -0.04 \text{ min}^{-1}$$

GIVING THE SOL'N

~~C~~ (

$$T(t) = 120 e^{-0.04t} + 70$$

Now FOR $110°$ WE HAVE

$$110 = 70 + 120 e^{-0.04t} \quad \Rightarrow \quad e^{-0.04t} = \frac{1}{3}$$

$$\Rightarrow \quad t = 27 \text{ min} \simeq 30 \text{ min}.$$