Topics in Mathematical Physics (PHYS 320): QPS 1 Spring 2009

This is the "problem set quiz" on our study of vector calculus. It is due on Wednesday, February 11. You may consult Boas, your Phys 320 class notes, and standard references such as the Schaums. Please cite any references as you use them. Do not consult the internet. Your solutions must be entirely your own work.

Ask questions when you have them. Try email: smajor@hamilton.edu or stop by my office.

Enjoy!

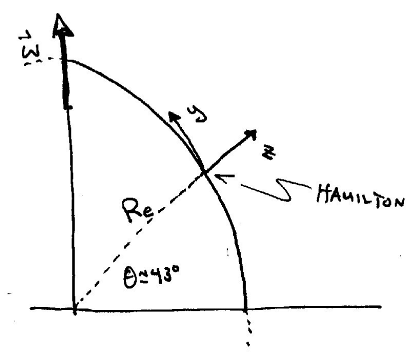
(1) For vectors fields $\mathbf{u}$ and $\mathbf{v}$ write $\mathbf{u} \times (\nabla \times \mathbf{v})$ with Levi-Civita symbols. Using any method you like, work out the vector identity.

(2) Find the Laplacian ∇^2 in cylindrical coordinates.

(3) Find the divergence of $\mathbf{u} = \mathbf{r}/r^3$ at $\mathbf{r} = 2\hat{\mathbf{e}}_\mathbf{x} + \hat{\mathbf{e}}_\mathbf{y} - \hat{\mathbf{e}}_\mathbf{z}$.

(4) Using the vector fields $\mathbf{u} = xy\hat{\mathbf{e}}_\mathbf{x} + y^2\hat{\mathbf{e}}_\mathbf{y} + z\hat{\mathbf{e}}_\mathbf{z}$ and $\mathbf{v} = x^2\hat{\mathbf{e}}_\mathbf{x} + xy\hat{\mathbf{e}}_\mathbf{y} + yz\hat{\mathbf{e}}_\mathbf{z}$, calculate $(\mathbf{u} \cdot \nabla)\mathbf{v}$.

(5) Another blast of arctic air wind, from due North, blows across the Hill (latitude 43^o) at 13 knots ≈ 6.7 m/s. Find the *magnitudes* of the Coriolis acceleration $2\omega \times \mathbf{v}$ and centripetal acceleration $\omega \times (\omega \times \mathbf{R})$ of a particle of this arctic air. For this problem let's use the coordinate system that has an origin fixed on the ground, a z-axis pointing away from the center of the earth ("up"), a x-axis pointing East, and a y-axis pointing North. Earth has an ω of about 7.3×10^{-5} rad s^{-1}. Here's a sketch of the earth showing the coordinate system.



The radius of earth is about 6400 km. Please give your numerical answer in SI units.

(6) Feeling chilly, you wish to travel towards a warmer place. The temperature is given by $T(x, y, z) = \alpha(x^2 + yz)$ (where α is a constant with units $^oC/m^2$). In what direction should you move?

(7) Congratulations! You have been appointed head of the new Hamilton Tokamak fusion research facility (fyi a dozen or so exist around the world). Realizing that the geometry of the plasma containment facility is a doughnut, bagel, or, mathematically speaking, a torus, you dust off your toroidal coordinate system (r, θ, ϕ). This orthogonal curvilinear coordinate system is defined via

$$x = R_o \cos\phi - r\cos\theta\cos\phi \quad (1)$$

$$y = R_o \sin\phi - r\cos\theta\sin\phi \quad (2)$$

$$z = r\sin\theta \quad (3)$$

where R_o, a constant, is the distance to the center of the torus. [Although it is not necessary for the solution to the problem, the coordinates (r, θ) are the usual polar coordinates in the plane perpendicular to the circle around the center of the torus. The position around this circle is given by ϕ.]

(a) Find ds^2.
(b) Find the scale factors h_r, h_θ, and h_ϕ.
(c) Write the gradient $\nabla\Phi$.
(d) What is the Laplacian ∇^2?
(e) EXTRA (only if you are enjoying this): What is the curl of a vector in these coordinates?

(8) Use the Helmholtz theorem to show that a "curl-free" (meaning $\nabla \times \mathbf{v} = 0$) vector field $\mathbf{v}$ is uniquely determined by a scalar field or potential (and the gradient of the scalar potential at the boundary). Hint: Use a vector identity we discussed in class, e.g. on Friday, January 30)