FROM

ARFKEN

In this section we develop a method of obtaining one solution of the linear, second-order, homogeneous differential equation. The method, a series expansion, will always work, provided the point of expansion is no worse than a regular singular point. In physics, this very gentle condition is almost always satisfied.

A *linear, second-order, homogeneous* differential equation may be put in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0. \tag{8.20}$$

The equation is *homogeneous* because each term contains $y(x)$ or a derivative; *linear* because each y, dy/dx, or d^2y/dx^2 appears as the first power—and no products. In this section we shall develop (at least) one solution of Eq. 8.20. In Section 8.5 we shall develop a second, independent solution and prove that no third, independent solution exists. Therefore, the most general solution of Eq. 8.20 may be written

$$y(x) = c_1y_1(x) + c_2y_2(x). \tag{8.20a}$$

Our physical problem may lead to a *nonhomogeneous*, linear, second-order differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x). \tag{8.20b}$$

The function on the right, $F(x)$, represents a source (such as electrostatic charge) or a driving force (as in a driven oscillator). Specific solutions of this nonhomogeneous equation are touched on in Ex. 8.5.19. They are explored in some detail, using Green's function techniques, in Sections 8.6, 16.5, and 16.6, and with a Laplace transform technique in Section 15.10. Calling this solution y_p, we may add to it any solution of the corresponding homogeneous equation (Eq. 8.20). Hence, the most general solution of Eq. 8.20b is

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x). \tag{8.20c}$$

The constants c_1 and c_2 will eventually be fixed by boundary conditions.

For the present, we assume that $F(x) = 0$, that our differential equation is homogeneous. We shall attempt to develop a solution of our linear, second-order, homogeneous differential equation, Eq. 8.20, by substituting in a power series with undetermined coefficients. Also available as a parameter is the power of the lowest nonvanishing term of the series. To illustrate, we apply the method to two important differential equations. First the linear oscillator equation

$$\frac{d^2y}{dx^2} + \omega^2y = 0, \tag{8.21}$$

with known solutions $y = \sin \omega x, \cos \omega x$.

We try

$$\begin{aligned} y(x) &= x^k(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) \\ &= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \qquad a_0 \neq 0, \end{aligned} \tag{8.22}$$

with the exponent k and all the coefficients a_λ still undetermined. By differentiating twice, we obtain

$$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1},$$

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}.$$

By substituting into Eq. 8.21, we have

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0. \tag{8.23}$$

From our analysis of the uniqueness of power series (Chapter 5) the coefficients of each power of x on the left-hand side of Eq. 8.23 must vanish individually.

The lowest power of x appearing in Eq. 8.23 is x^{k-2}, for $\lambda = 0$ in the first summation. The requirement that the coefficient vanish[1] yields

$$a_0 k(k-1) = 0.$$

We had chosen a_0 as the coefficient of the lowest nonvanishing terms of the series (Eq. 8.22), hence, by definition, $a_0 \neq 0$. Therefore we have

$$k(k-1) = 0. \tag{8.24}$$

This equation, coming from the coefficient of the lowest power of x, we call the *indicial equation*. The indicial equation and its roots are of critical importance to our analysis. Clearly, in this example we must require either that $k = 0$ or $k = 1$.

Before considering these two possibilities for k, we return to Eq. 8.23 and demand that the remaining net coefficients, say the coefficient of x^{k+j} ($j \geqslant 0$), vanish. We set $\lambda = j + 2$ in the first summation and $\lambda = j$ in the second. (They are independent summations and λ is a dummy index.) This results in

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0$$

or

$$a_{j+2} = -a_j \frac{\omega^2}{(k+j+2)(k+j+1)}. \tag{8.25}$$

This is a two-term *recurrence relation*. Given a_j, we may compute a_{j+2} and then a_{j+4}, a_{j+6}, and so on up as far as desired. The reader will note that for this example, if we start with a_0, Eq. 8.25 leads to the even coefficients a_2, a_4, etc., and ignores a_1, a_3, a_5, etc. Since a_1 is arbitrary, let us set it equal to zero (cf. Ex. 8.4.2 and 8.4.3) and then by Eq. 8.25

$$a_3 = a_5 = a_7 = \cdots = 0,$$

and all the odd power coefficients vanish. Do not worry about the lost odd powers; the object here is to get a solution. The rejected odd powers will actually reappear when the *second* root of the indicial equation is used.

Returning to Eq. 8.24, our indicial equation, we first try the solution $k = 0$.

[1] Uniqueness of power series, Section 5.7.

The recurrence relation (Eq. 8.25) becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)}, \tag{8.26}$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0, \text{ etc.}$$

By inspection (and mathematical induction)

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0. \tag{8.27}$$

and our solution is

$$y(x)_{k=0} = a_0\left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots\right]$$

$$= a_0 \cos \omega x. \tag{8.28}$$

If we choose the indicial equation root $k = 1$ (Eq. 8.25) the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}. \tag{8.29}$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \text{ etc.}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0. \tag{8.30}$$

For this choice, $k = 1$, we obtain

$$y(x)_{k=1} = a_0 x\left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots\right]$$

$$= \frac{a_0}{\omega}\left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots\right]$$

$$= \frac{a_0}{\omega} \sin \omega x. \tag{8.31}$$

To summarize this approach, Eq. 8.23 may be written schematically as shown in Fig. 8.1. From the uniqueness of power series (Section 5.7), the total coefficient of each power of x must vanish—all by itself. The requirement that the first coefficient (I) vanish leads to the indicial equation, Eq. 8.24. The second coefficient is handled by setting $a_1 = 0$. The vanishing of the coefficient of x^k (and higher powers, taken one at a time) leads to the recurrence relation Eq. 8.25.

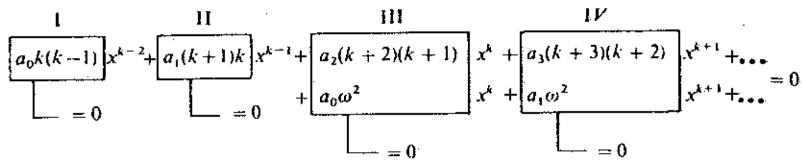

FIG. 8.1

This series substitution, known as Frobenius' method, has given us two series solutions of the linear oscillator equation. However, there are two points about such series solutions that must be strongly emphasized:

1. The series solution should always be substituted back into the differential equation, to see if it works, as a precaution against algebraic and logical errors. Conversely, if it works, it is a solution.

2. The acceptability of a series solution depends on its convergence (including asymptotic convergence). It is quite possible for Frobenius' method to give a series solution which satisfies the original differential equation when substituted in but which does *not* converge over the region of interest. Legendre's differential equation illustrates this situation.

The alert reader will note that we obtained one solution of even symmetry, $y_1(x) = y_1(-x)$, and one of odd symmetry, $y_2(x) = -y_2(-x)$. This is not just an accident but a direct consequence of the form of the differential equation. Writing a general differential equation as

$$\mathscr{L}(x)\, y(x) = 0, \tag{8.32}$$

in which $\mathscr{L}(x)$ is the differential operator, we see that for the linear oscillator equation (Eq. 8.21) $\mathscr{L}(x)$ is even, that is,

$$\mathscr{L}(x) = \mathscr{L}(-x). \tag{8.33}$$

Often this is described as even parity.

Whenever the differential operator has a specific parity or symmetry, either even or odd, we may interchange $+x$ and $-x$, and Eq. 8.32 becomes

$$\pm\mathscr{L}(x)\, y(-x) = 0, \tag{8.34}$$

$+$ if $\mathscr{L}(x)$ is even, $-$ if $\mathscr{L}(x)$ is odd. Clearly, if $y(x)$ is a solution of the differential equation, $y(-x)$ is also a solution. Then any solution may be resolved into even and odd parts,

$$y(x) = \tfrac{1}{2}[y(x) + y(-x)] + \tfrac{1}{2}[y(x) - y(-x)], \tag{8.35}$$

the first bracket on the right giving an even solution, the second an odd solution.

If we refer back to Section 8.3, we can see that Legendre, Chebyshev, Bessel, simple harmonic oscillator, and Hermite equations (or differential operators) all exhibit this even parity. Solutions of all of them may be presented as series of even powers of x and separate series of odd powers of x. The Laguerre differential operator has neither even nor odd symmetry; hence its solutions cannot be expected to exhibit even or odd parity.

Limitations of series approach—Bessel's equation. This attack on the linear oscillator equation was perhaps a bit too easy. By substituting the power series (Eq. 8.22) into the differential equation (Eq. 8.21) we obtained two independent solutions with no trouble at all.

To get some idea of what can happen we try to solve Bessel's equation,

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{8.36}$$

using y' for dy/dx and y'' for d^2y/dx^2. Again assuming a solution of the form

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda},$$

we differentiate and substitute into Eq. 8.36. The result is

$$\begin{aligned}\sum_{\lambda=0}^{\infty} a_\lambda(k+\lambda)(k+\lambda-1)x^{k+\lambda} &+ \sum_{\lambda=0}^{\infty} a_\lambda(k+\lambda)x^{k+\lambda} \\ &+ \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda} = 0.\end{aligned} \tag{8.37}$$

By setting $\lambda = 0$, the coefficient of x^k, the lowest power of x appearing on the left-hand side, is

$$a_0[k(k-1) + k - n^2] = 0, \tag{8.38}$$

and again $a_0 \neq 0$ by definition. Equation 8.38 therefore yields the *indicial equation*

$$k^2 - n^2 = 0 \tag{8.39}$$

with solutions $k = \pm n$.

It is of some interest to examine the coefficient of x^{k+1} also. Here we obtain

$$a_1[(k+1)k + k + 1 - n^2] = 0$$

or

$$a_1(k+1-n)(k+1+n) = 0. \tag{8.40}$$

For $k = \pm n$ neither $k+1-n$ nor $k+1+n$ vanishes and we *must* require $a_1 = 0$.[1]

Proceeding to the coefficient of x^{k+j} for $k = n$, we set $\lambda = j$ in the first, second, and fourth terms of Eq. 8.37 and $\lambda = j - 2$ in the third term. By requiring the resultant

[1] $k = \pm n = -\frac{1}{2}$ are exceptions.

coefficient of x^{k+j} to vanish we obtain

$$a_j[(n+j)(n+j-1)+(n+j)-n^2]+a_{j-2}=0.$$

When j is replaced by $j+2$, this can be rewritten

$$a_{j+2}=-a_j\frac{1}{(j+2)(2n+j+2)}, \tag{8.41}$$

which is the desired recurrence relation. Repeated application of this recurrence relation leads to

$$a_2=-a_0\frac{1}{2(2n+2)}=-\frac{a_0 n!}{2^2 1!(n+1)!},$$

$$a_4=-a_2\frac{1}{4(2n+4)}=+\frac{a_0 n!}{2^4 2!(n+2)!},$$

$$a_6=-a_4\frac{1}{6(2n+6)}=-\frac{a_0 n!}{2^6 3!(n+3)!}, \text{ etc.,}$$

and in general

$$a_{2p}=(-1)^p\frac{a_0\, n!}{2^{2p}p!(n+p)!}. \tag{8.42}$$

Inserting these coefficients in our assumed series solution, we have

$$y(x)=a_0x^n\left[1-\frac{n!x^2}{2^2 1!(n+1)!}+\frac{n!x^4}{2^4 2!(n+2)!}-\cdots\right]. \tag{8.43}$$

In summation form

$$y(x)=a_0\sum_{j=0}^{\infty}(-1)^j\frac{n!x^{n+2j}}{2^{2j}j!(n+j)!}$$

$$=a_0 2^n\sum_{j=0}^{\infty}(-1)^j\frac{n!}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j} \tag{8.44}$$

In Chapter 11 the final summation is identified as the Bessel function $J_n(x)$. Notice that this solution $J_n(x)$ has either even or odd symmetry[1] as might be expected from the form of Bessel's equation.

When $k=-n$ and n is not an integer, we may generate a second distinct series to be labeled $J_{-n}(x)$. However, when $-n$ is a negative integer, trouble develops. The recurrence relation for the coefficients a_j is still given by Eq. 8.41, but with $2n$ replaced by $-2n$. Then, when $j+2=2n$ or $j=2(n-1)$, the coefficient a_{j+2} blows up and we have no series solution. This catastrophe can be remedied in Eq. 8.44, as it is done in Chapter 11, with the result that

$$J_{-n}(x)=(-1)^n J_n(x), \qquad n \text{ an integer.} \tag{8.45}$$

[1] $J_n(x)$ is an even function if n is an even integer, an odd function if n is an odd integer. For nonintegral n the x^n has no such simple symmetry.

The second solution simply reproduces the first. We have failed to construct a second independent solution for Bessel's equation by this series technique when n is an integer.

By substituting in an infinite series, we have obtained two solutions for the linear oscillator equation and one for Bessel's equation (two if n is not an integer). To the questions "Can we always do this? Will this method always work?", the answer is no, we cannot always do this. This method of series solution will not always work.

The success of the series substitution method depends on the roots of the indicial equation and the degree of singularity of the coefficients in the differential equation. To understand better the effect of the equation coefficients on this naïve series substitution approach consider four simple equations

$$y'' - \frac{6}{x^2} y = 0, \tag{8.46a}$$

$$y'' - \frac{6}{x^3} y = 0, \tag{8.46b}$$

$$y'' + \frac{1}{x} y' - \frac{a^2}{x^2} y = 0, \tag{8.46c}$$

$$y'' + \frac{1}{x^2} y' - \frac{a^2}{x^2} y = 0. \tag{8.46d}$$

The reader may show easily that for Eq. 8.46a the indicial equation is

$$k^2 - k - 6 = 0,$$

giving $k = 3, -2$. Since the equation is homogeneous in x (counting d^2/dx^2 as x^{-2}), there is no recurrence relation; $a_i = 0$ for $i > 0$. However, we are left with two perfectly good solutions, x^3 and x^{-2}.

Equation 8.46b differs from 8.46a by only one power of x, but this sends the indicial equation to

$$-6a_0 = 0,$$

with no solution at all, for we have agreed that $a_0 \neq 0$. Our series substitution worked for Eq. 8.46a, which had only a regular singularity, but broke down at Eq. 8.46b, which has an irregular singular point at the origin.

Continuing with Eq. 8.46c, we have added a term y'/x. The indicial equation is

$$k^2 - a^2 = 0,$$

but again there is no recurrence relation. The solutions are $y = x^a,\ x^{-a}$, both perfectly acceptable one term series.

When we change the power of x in the coefficient of y' from -1 to -2, Eq. 8.46d, there is a drastic change in the solution. The indicial equation (with only the y' term contributing) becomes

$$k = 0.$$

There is a recurrence relation

$$a_{j+1} = -a_j \frac{a^2 - j(j-1)}{j+1}.$$

Unless the parameter a is selected to make the series terminate, we have

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty}\frac{j(j-1)}{j+1}$$

$$= \lim_{j\to\infty}\frac{j^2}{j} = \infty .$$

Hence our series solution diverges for all $x \neq 0$. Again our method worked for Eq. 8.46*c* with a regular singularity but failed when we had the irregular singularity of 8.46*d*.

Fuchs's theorem. The answer to the basic question when the method of series substitution can be expected to work is given by Fuchs's theorem, which asserts that we can always obtain at least one power series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point. If we attempt an expansion about an irregular or essential singularity, our method may fail as it did for Eqs. 8.46*b* and 8.46*d*. Fortunately, the more important equations of mathematical physics listed in Section 8.3 have no irregular singularities in the finite plane. Further discussion of Fuchs's theorem appears in Section 8.5.

From Table 8.3, Section 8.3, infinity is seen to be a singular point for all of the equations considered. As a further illustration of Fuchs's theorem, Legendre's equation (with infinity as a regular singularity) has a convergent series solution in negative powers of the argument (Section 12.10). In contrast, Bessel's equation (with an irregular singularity at infinity) yields asymptotic series (Sections 5.11 and 11.6). While extremely useful, these asymptotic solutions are technically divergent.

Summary. If we are expanding about an ordinary point or at worst about a regular singularity, the series substitution approach will yield at least one solution (Fuchs's theorem).

Whether we get one or two distinct solutions depends on the roots of the indicial equation.

1. If the two roots of the indicial equation are equal, we can obtain only one solution by this series substitution method.
2. If the two roots differ by a nonintegral number, two independent solutions may be obtained.
3. If the two roots differ by an integer, the larger of the two will yield a solution. The smaller may or may not give a solution, depending on the behavior of the coefficients. In the linear oscillator equation we obtain two solutions; for Bessel's equation, only one solution.

EXERCISES

8.4.1 A series solution of Eq. 8.20 is attempted, expanding about the point $x = x_0$. If x_0 is an ordinary point show that the indicial equation has roots $k = 0, 1$.