In which we play with metrics, derive some Einstein equations and explore warp drives.

Reading:
Since PS 4 we have discussed sections 11.1 - 11.2 and 12.1 - 12.4. In our last week we return to black holes and study rotation, and perhaps the Hawking effect, all in Chapter 11.

Problems:
All numbered problems are from Schutz. Problems 2 and 6 are optional.

(1) The tidal effects of space-time curvature is given by geodesic deviation. For the worldline given in 11.21 find the minimum mass a Schwarzschild black hole must have so that an in-falling star is not torn apart before it crosses the horizon. The actual process is complicated but for this problem assume that the acceleration gradient for a healthy star is 0.5 m s$^{-2}$ per m.

(2) Optional 1 point An observer named Pynchon decides to explore the geometry outside a Schwarzschild black hole of mass $M$ by starting with an initial velocity in asymptopia, falling in towards the black hole and then returning to asymptopia. What is the closest approach that Pynchon can make to the horizon? How can Pynchon arrange to have a long time to study the geometry near this smallest radius?

(3) Find the Einstein equations, with cosmological constant, for the cosmological metric

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right)$$

I recommend using the mathematical notebook “einstein.nb” to derive these, although any method of computation is fine.

(4) In 1995 Alcubierre found the warp drive metric

$$ds^2 = -dt^2 + [dx - V_s(t)f_R(r_s)dt]^2 + dy^2 + dz^2$$

where the ship’s velocity and radius are given by

$$V_s(t) = \frac{dx_s(t)}{dt} \text{ and } r_s = \sqrt{(x - x_s(t))^2 + y^2 + z^2}.$$ 

The “warp drive shape” function $f_R(r_s)$ is a smooth, positive function that satisfies $f_R(0) = 1$ and decreases away from the origin to vanish at some $r_s > R$. The (faster than light) trajectory is given by the worldline $x_s(t)$. This allows a total travel time $T$ to be less that the total distance traveled $D$ in the above metric. Let’s check this metric out:

(a) Find the spatial interval $dS^2$ on a spatial section where $dt = 0$ and show that it is flat. (Nothing odd there!)

(b) Find the light cone structure in the $t - x$ space-time. To be concrete you can assume that the warp drive shape is

$$f_R(r_s) = 1 - \left( \frac{r_s}{R} \right)^4$$

Sketch the light cones in and around the ship and comment on your results.

(c) Show that at every point along $x_s(t)$ the 4-velocity lies inside the forward light cone, i.e. inside the ship there is no funny business.

(5) In the warp drive spacetime, find how much ship time elapses on a trip lasting a time $T$. 

(6) **Optional 1 point** So why don’t we just build a warp drive? Modify the mathematica “einstein” notebook to compute the energy-momentum tensor for the warp drive spacetime. Show that the components normal to a surface of constant $t$ are

$$-rac{1}{8\pi} \frac{V_s^2(y^2 + z^2)}{4r_s^2} \left( \frac{df}{dr_s} \right)^2$$

i.e. **negative**. No stuff with which we build things (metal, fields, even Krispy Creme donuts) has negative energy density - oh well.

(7) Adjusting the cosmological parameters at will play with the Friedmann equation evolution notebook and print out your favorite history of the universe. Record $H_0, \Omega_r, \Omega_\Lambda, \Omega_m, \text{and } \Omega_c$.

(8) 12.17, which sets up the next problem.

(9) Show that, in the context of homogeneous and isotropic cosmologies, if $\rho + 3p$ is always positive there will be a big bang in the past. Do this by first differentiating the Friedmann equation with the effective potential to obtain

$$\ddot{a} + \frac{4\pi}{3} (\rho + 3p)a = 0.$$ 

(SHM!) Argue from this that there will be crunches, or the big bang and the gnab gib, in the future and past.