Classical Mechanics (PHYS 350): Final Practice

Fall 2011 v1.0

Here is a potpourri of questions, some from old exams. Problems 1-7 are from practice finals plus a couple new questions. Problems 8-12 are from timed finals. Note that the actual final will not require solutions to 12 problems!!

Instructions: "This final, held under the auspices of the Hamilton Honor Code, consists of 5 problems in three hours. Please do not consult with other resources. Your solutions must represent your work only. Please ask questions whenever the statement of the problem is puzzling."

Please use one page (or more) per solution.

Problems:

(1) Derive the Hamiltonian equations of motion. Demonstrate their use by solving the simple pendulum using the Hamiltonian method.

(2) Derive the Euler equations for rigid body rotation.

(3) Starting from a variational principle of your choice, derive general equations of motion.

(4) What is an effective potential? Give an example.

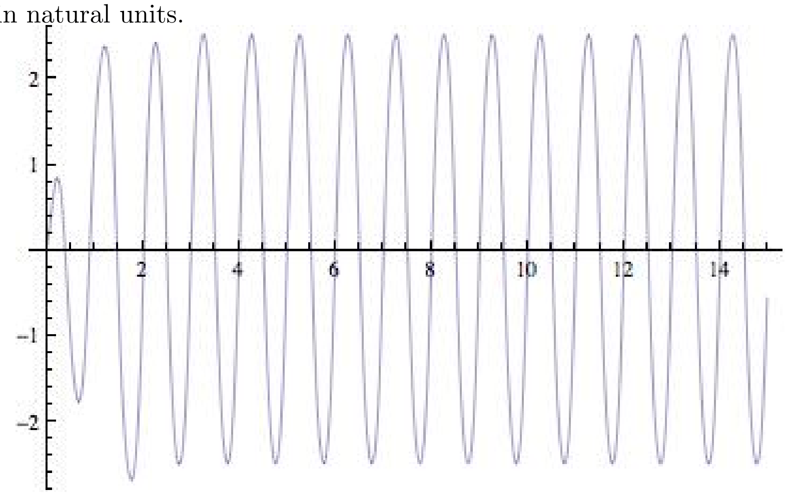
(5) A chain of mass M and length l is suspended vertically with its lowest end touching a scale. The chain is released from rest and falls onto the scale. (a) What is the reading on the scale when a length x has fallen? Assume a constant linear mass density. (b) What is the maximum reading? Is it larger or less than the weight of the chain Mg? Why?

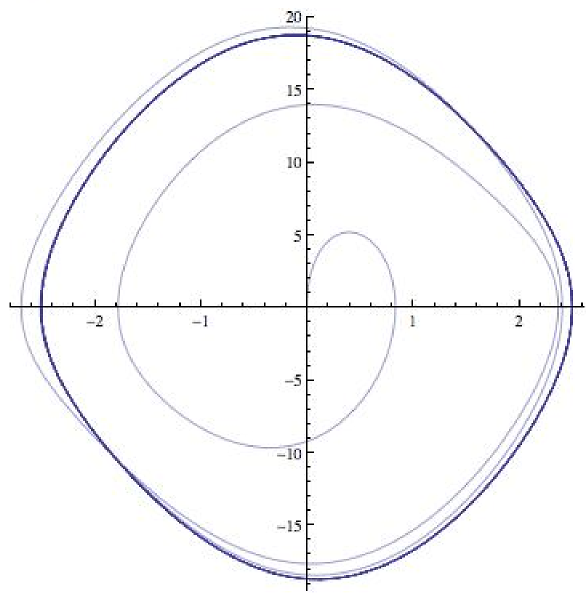
(6) A thin hoop of radius R and mass M oscillates in its own plane with one point on the hoop fixed. A bead of mass m is free to move among this hoop.
   (a) Find the moment of inertia of the hoop.
   (b) Using the angles in the diagram find the kinetic and potential energies.
   (c) Let the masses be equal, $M = m$, and make a small angle approximation. Show that the Lagrangian becomes
$$L = \frac{1}{2}mR^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) - \frac{gmR}{2}(2\theta^2 + \phi^2).$$
   (d) Compute the equations of motion
   (e) Finally, find the angular frequencies of the normal modes for this system.

(7) Suppose that in a far and distant planetary system planets are observed to follow a spiral orbit $r = c\theta^2$. (c is a constant with dimensions of length.) What new gravitational force and potential are at play in this strange planetary system? Sketch the potential.

(8) **Non-linear Pendulum** At last! You have working code for exploring the dynamics of non-linear systems, including a damped, driven pendulum. You choose a dimensionless driving amplitude γ as a parameter and begin exploring.

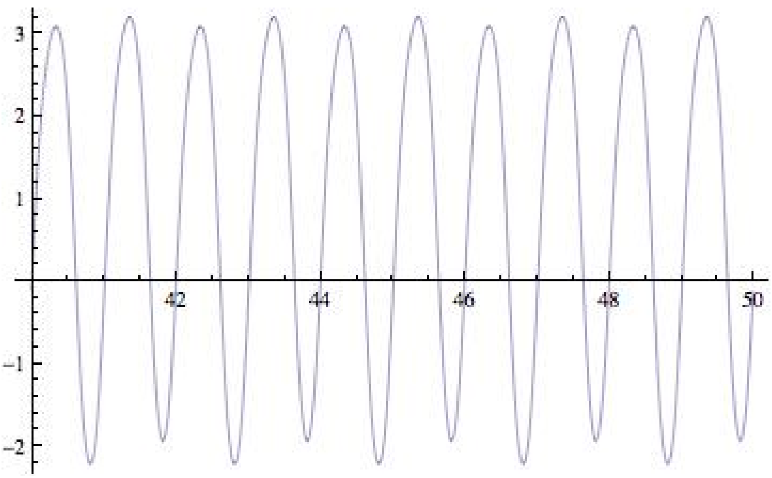
(a) At $\gamma = 0.9$ you find the following plots of ϕ vs. t and the phase space trajectory for $t = 0$ to $t = 15$ in natural units.

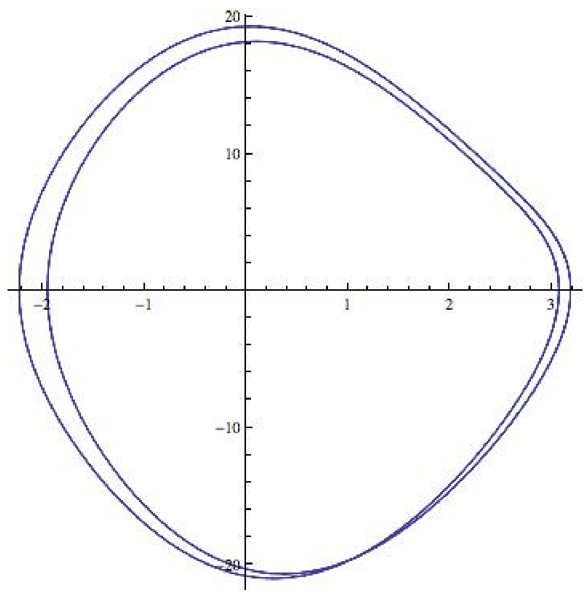




Briefly describe the motion.

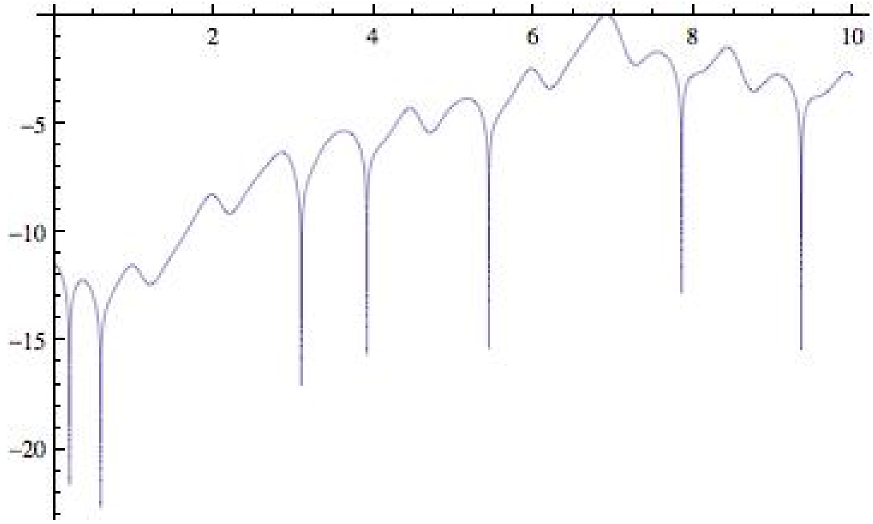
(b) At $\gamma = 1.073$ you make the following plots





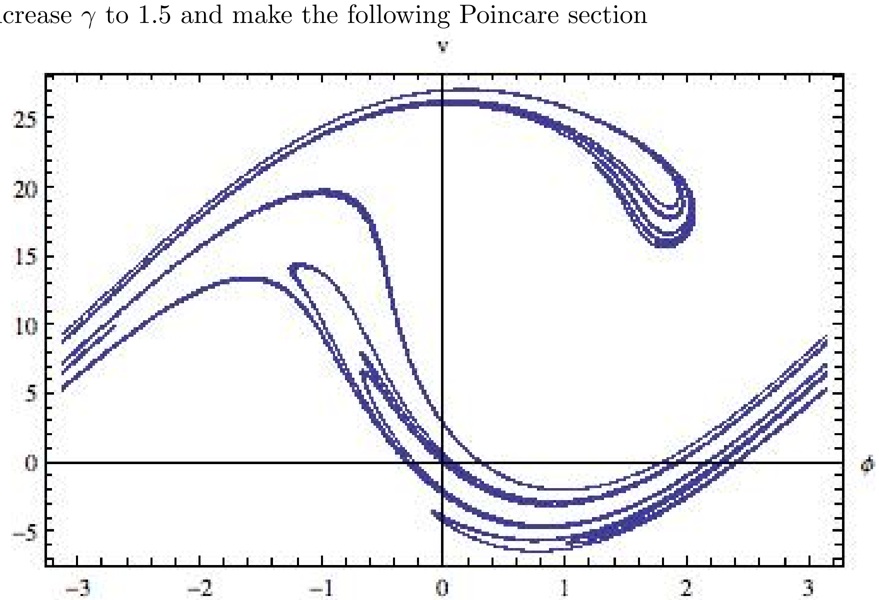
Note that you cleverly plot the motion at "late times" to remove the transients. Briefly describe the motion.

(c) After playing around with the parameter for some time you make the following plot for $\gamma = 1.084$



This is the log of the absolute value of the difference of two trajectories that differed in initial angles by a part in 10^4. What is the utility of such a plot and what does it tell you?

(d) You increase γ to 1.5 and make the following Poincare section



What is this plot and what does it tell you about the system? ("v" is the momentum.)

(9) **Bead on Loop** A frictionless bead of mass m is confined to a rotating circular loop or radius R. The axis of rotation is in the plane of the loop and is displaced and distance b from the diameter. The loop is driven at a constant angular speed Ω. TPlease use the angle defined as displacement from the line along the diameter and on the side where the axis is placed.

(a) Find the Lagrangian and the equation of motion for the bead.

(b) Find the equilibrium points.

(c) For small angles this system has the form of an oscillator with time-dependent frequency. Comment on the stability of oscillations as b is varied from 0 to R.

(10) **Spherical Pendulum** A spherical pendulum consists of mass m attached to an inflexible rod of negligible mass. The upper end of the rod can rotate freely in all directions about one end.

(a) How many degrees of freedom does the system have?
(b) Find the Hamiltonian in spherical coordinates.
(c) Find the constant of motion, calling it ℓ, combine this term with the potential to define an effective potential $U_{eff}(\theta)$.
(d) Sketch U_{eff} as a function of θ.
(e) Briefly discuss the motion for $\ell = 0$ and $\ell \neq 0$.

(11) **Shopping Mania** The tallest elevator shaft in the Empire State building is 1454 ft. In a fit of extreme last minute shopping mania (ELMSM) you drop a case of Zhu Zhu Pets from the top (carefully clearing the way prior to the drop). Where does the case of pets land? Please use coordinates such that you drop the case from (x, y, z) =(0,0,1454 ft). Ignore the effects of air resistance.The latitude of NYC is 41° N. 1 ft. = 0.3048 m.

(12) **Foucault pendulum** A Foucault pendulum hangs in the North Atrium. It is a long (17 m) pendulum that is well within the small angle approximation.
(a) Find the Lagrangian for the pendulum.
(b) Find the equations of motion.
(c) Use these equations to find the period for the rotational motion.
(d) Determine how many degrees should the pendulum precess in 1 hour.

1. Potentially useful odds and ends:

1 ft. = 0.3048 m. The Earth rotates with an angular speed of about 7.3×10^{-5} rad/s.

In spherical coordinates: $ds = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$

Approximations

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots \text{ and } (1+x)^n \simeq 1 + nx + \frac{n(n-1)}{2}x^2$$

Trig idents

$$\sin(a \pm b) = \sin a \, \cos b \pm \cos a \, \sin b$$
$$\cos(a \pm b) = \cos a \, \cos b \mp \sin a \, \sin b$$
$$\sin 2\theta = 2\sin\theta\cos\theta \text{ and } \cos 2\theta = \cos^2\theta - \sin^2\theta$$

Gravitational field

$$G = 6.67 \times 10^{-11}\mathrm{N}kg^{-2}m^2$$
$$\mathbf{g} = -\nabla\Phi$$
$$U(r) = -\frac{GMm}{r}$$

For central potential motion

$$L(\phi, \dot{\phi}) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

The magical u equation is

$$u'' + u = -\frac{\mu}{u^2\ell^2}F(1/u)$$

(Generalized) momenta

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Hamilton mechanics

$$H = H(q_k, p_k, t) = \sum_k p_k\dot{q}_k - L(q_i, \dot{q}_i, t)$$

with eom

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Coupled Oscillations:

$$(K - \omega_I^2 M)\mathbf{a} = 0$$

Rigid Body Mechanics:

$$I_{ij} = \int \rho\left(r^2\delta_{ij} - r_ir_j\right)dV$$

$$T = \frac{1}{2}\omega^T I\omega \text{ or } T = \frac{1}{2}\sum_i I_i\omega_i^2 \text{ while } L_i = I_{ij}\omega_j$$

Euler's equations with torques:

$$I_1\omega_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$
$$I_1\omega_2 - (I_3 - I_1)\omega_3\omega_1 = N_2$$
$$I_1\omega_3 - (I_1 - I_2)\omega_1\omega_1 = N_3$$

$$\begin{aligned}\omega_1 &= \dot\phi\sin\theta\sin\psi+\dot\theta\cos\psi\\ \omega_2 &= \dot\phi\sin\theta\cos\psi-\dot\theta\sin\psi\\ \omega_3 &= \dot\phi\cos\theta+\dot\psi\end{aligned}$$

For damped, driven oscillation:

$$\ddot x+2\beta\dot x+\omega_o^2x=A\cos\omega t$$
$$\omega_R^2=\omega_o^2-2\beta$$
$$Q:=\frac{\omega_R}{2\beta}\approx\frac{\omega_o}{2\beta}$$
$$E=E_oe^{-2\beta t}$$

The relation between acceleration in non-inertial frames (primed) and inertial frames (un-primed) is

$$\mathbf{a}=\mathbf{A}+\mathbf{a}'+\dot\omega\times\mathbf{r}'+\omega\times(\omega\times\mathbf{r}')+\mathbf{2}\omega\times\mathbf{v}'.$$

For a general vector $\mathbf{Q}$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathrm{IN}}=\left(\frac{d\mathbf{Q}'}{dt}\right)_{\mathrm{Non-IN}}+\omega\times\mathbf{Q}$$