The Attractiveness of Loops and Ribbons

There would appear to be strong reasons for believing that the continuum concept may eventually have to be abandoned as one of the basic ingredients of a fundamental physical theory.

– R. Penrose, *Theory of Quantized Directions*

Seth Major

SIAM Colloquium RPI

4 February 2008

Outline

(Some of) the story so far from loop quantum gravity

- How can a graph represent geometry?
- Are such graphs related to gravity?
- Adding a cosmological constant
- Possible observational consequences

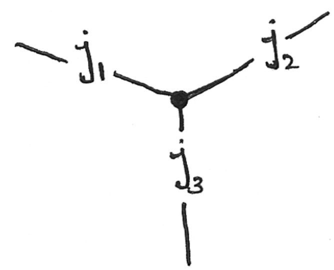
Spin networks

Graphs for 3 dimensional spatial geometry

Edges A finite set $\{e_i\}$ embedded and labeled with "spin"

$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, ...$$

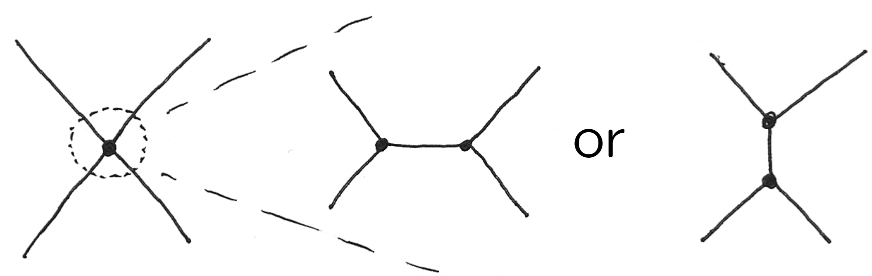
Vertices finite set $\{v_i\}$ Trivalent vertex spins satisfy the triangle inequalities



$$j_1 + j_2 \geq j_3, \quad j_2 + j_3 \geq j_1, \quad j_1 + j_3 \geq j_2$$

and the sum $j_1 + j_2 + j_3$ is an integer.

Higher valence vertices are labeled with "intertwiners" - decomposition in terms of trivalent vertices



These are **spin networks** (Penrose) states of geometry

Primer on Quantum Mechanics

(i) Complex vector space of "states" $\mid \psi \rangle$ with scalar product (Hilbert space)
In basis $\{\mid i \rangle\}$

$$\mid \psi \rangle = \sum_i \psi_i \mid i \rangle$$

Normed states - "The system exists in some state"

$$\sum_i |\langle i \mid \psi \rangle|^2 = 1$$

(ii) Physical measurements are represented by self-adjoint operators $\hat{A}$

$$\langle i \mid \hat{A}^\dagger \mid j \rangle \equiv \langle j \mid \hat{A} \mid i \rangle^* = \langle i \mid \hat{A} \mid j \rangle$$

Results of measurement are eigenvalues of self-adjoint operators (no degeneracy)

$$\hat{A} \mid \psi \rangle = a \mid \psi \rangle \text{ with } \mathrm{Prob}(A = a_i | \psi) = |\psi_i|^2$$

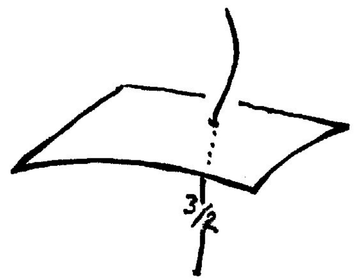
(iii) System evolves in time according to Schödinger's equation

$$i\hbar \frac{d}{dt} \mid s \rangle = \hat{H} \mid s \rangle$$

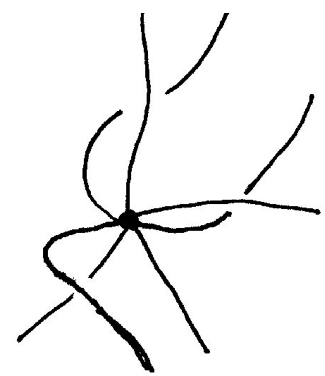
Discrete Spatial Geometry

Familiar geometric quantities arise through measurement - operators - on the spin network states

Area: spin network lines are "flux lines of area"



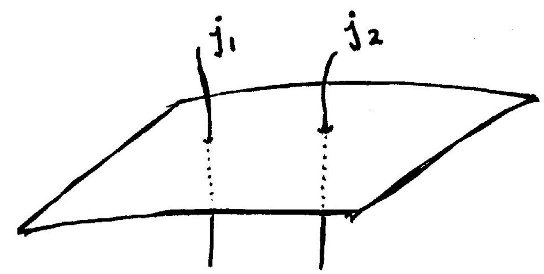
Volume arises only at vertices
Angle is defined at vertices



Discrete spectra for Geometry

Area*: $\widehat{A}_S \mid s\rangle = a \mid s\rangle$

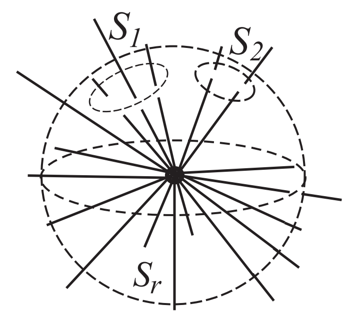
$$a = \ell_P^2 \sum_{n=1}^{N} \sqrt{j_n(j_n+1)} \text{ with}$$



$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} = 10^{-35} m$$

Angle†: $\widehat{\theta} \mid s\rangle = \theta \mid s\rangle$

$$\theta = \arccos\left(\frac{j_r(j_r+1) - j_1(j_1+1) - j_2(j_2+1)}{2\,[j_1(j_1+1)\,j_2(j_2+1)]^{1/2}}\right)$$

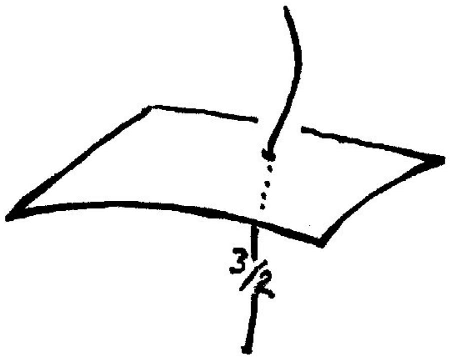


* Rovelli,Smolin Nuc. Phys. B **422** (1995) 593; Asktekar,Lewandowski Class. Quant. Grav. **14** (1997) A43
† SM Class. Quant. Grav. **16** (1999) 3859

Discrete spectra for Geometry

Area: $\widehat{A}_S \mid s\rangle = a \mid s\rangle$

Suppose a single edge, with $j = 3/2$, passes through a surface



A measurement of area would yield

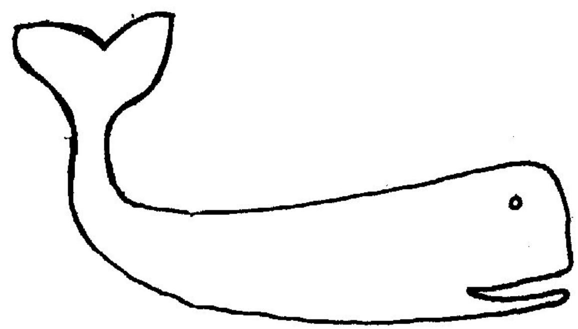
$$a = \ell_P^2 \sqrt{j(j+1)} = \ell_P^2 \frac{\sqrt{15}}{2} \approx 10^{-70} m^2$$

Small!

Discrete Spatial Geometry

What if the Planck length were $\ell_P = 2$ m ?

Suppose you observed a growing whale...

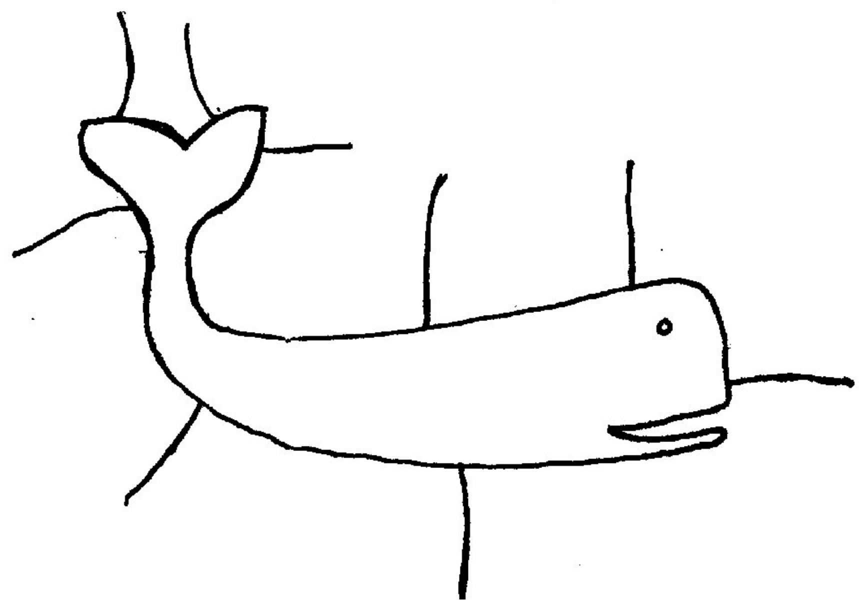


Discrete Spatial Geometry

What if the Planck length were $\ell_P = 2$ m ?

Suppose you observed a growing whale...
with surface area (all edges with $j = 1/2$)

$$a = \ell_P^2 \sum_{n=1}^{9} \sqrt{j_n(j_n+1)} = \ell_P^2 \, 9 \, \frac{\sqrt{3}}{2} \approx 31.2 \, m^2$$

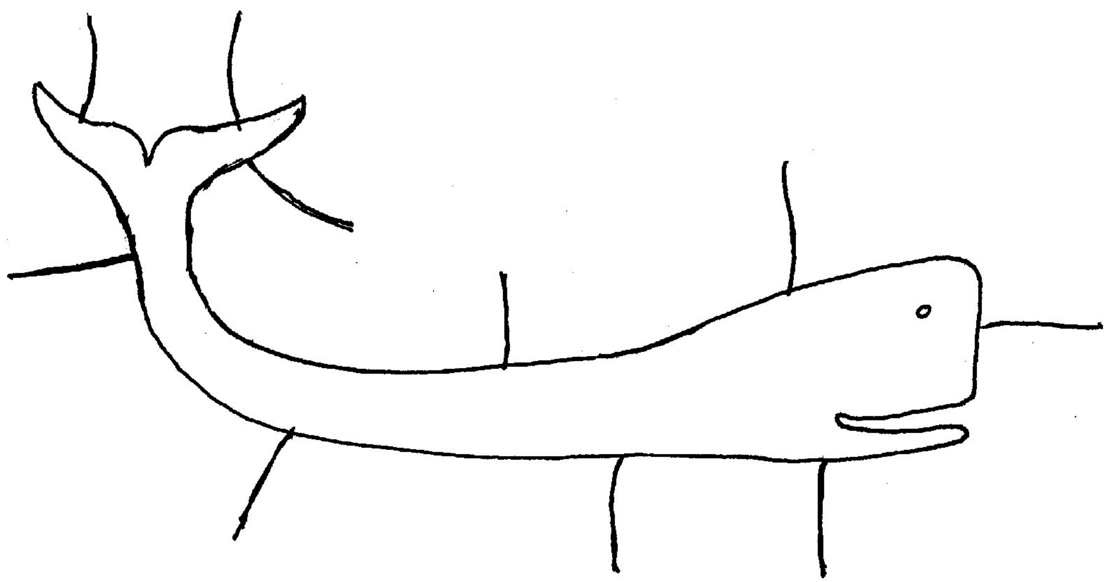


Discrete Spatial Geometry

What if the Planck length were $\ell_P = 2$ m ?

Suppose you observed a growing whale...
who grew the minimum amount to

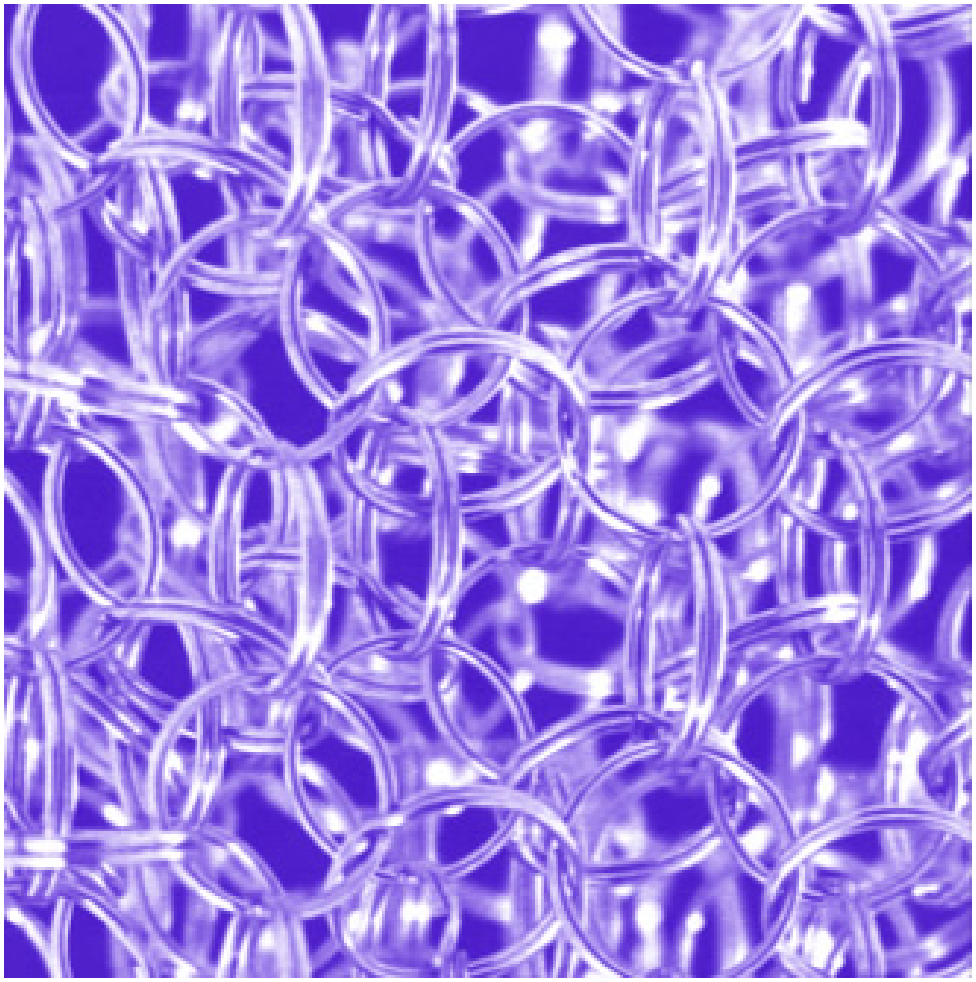
$$a = \ell_P^2 \sum_{n=1}^{10} \sqrt{j_n(j_n+1)} = \ell_P^2\, 10\, \frac{\sqrt{3}}{2} \approx 34.6\, m^2$$



Discrete Spatial Geometry

A link constructed of a set of $j = 1/2$ loops (key rings!).

What geometry does it have?



C. Rovelli, Physics World Nov. 2003

Attractiveness of loops?

In Einstein's general relativity the attractiveness of gravity arises through curvature.

Gravity: Einstein described the universe (a smooth manifold $M = S \times \mathbb{R}$) with a metric g_{ab} satisfying the non-linear partial differential equation (a,b,... = 0,1,2,3)

$$G_{ab} = 8\pi G\, T_{ab}$$

- **Spatial part** on S: $G_{ij} = 8\pi G\, T_{ij}$ Solutions give possible spatial geometry - "Space can curve".

Matter and light follow geodesics in the curved geometry $\rightsquigarrow$ attraction of gravity

- **Temporal part** on $\mathbb{R}$ $G_{0i} = 8\pi G\, T_{0i}$ tells us how space evolves

This talk will focus on vacuum solutions without cosmological constant

$$G_{ab} = 0$$

and with cosmological constant

$$G_{ab} + \Lambda g_{ab} = 0$$

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Finding quantum states of

$$\text{“}G_{ab} = 0\text{”}$$

is hard! It is easier with variables other than the (spatial) metric (and extrinsic curvature).

Attractiveness of loops?

General relativity with new variables (A, E).

[A is a smooth connection on principal su(2) bundle. E is a weighted frame field.]

Einstein equations in the new variables:

- **Spatial part** "$G_{ij} = 0$" solutions give possible spatial geometry. ⇒

$$D_a E^a = 0$$
$$E^b F_{ab} = 0$$

- **Temporal part** "$G_{0i} = 0$" tells us how space evolves ⇒

$$E^a \cdot (E^b \times F_{ab} + \frac{\Lambda}{6} \epsilon_{abc} E^b \times E^c) + \ldots = 0$$

- We can find solutions to the spatial part!

Attractiveness of loops?

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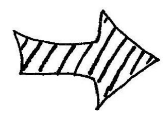
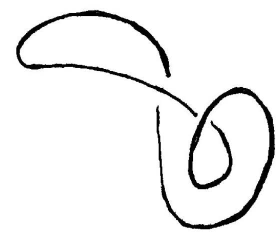


$$\begin{aligned} D_a E^a &= 0 \\ E^b F_{ab} &= 0 \end{aligned}$$

"$D_a E^a = 0$" (gauge freedom) divergence-free vector field $\rightsquigarrow$ loops!



A linear combination of these constraints imply that the state is invariant under smooth, invertible maps of space into itself - diffeomorphisms!

Attractiveness of loops?

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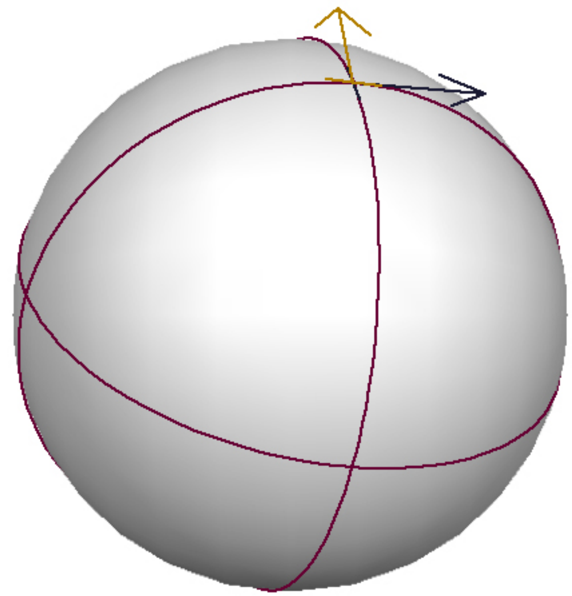


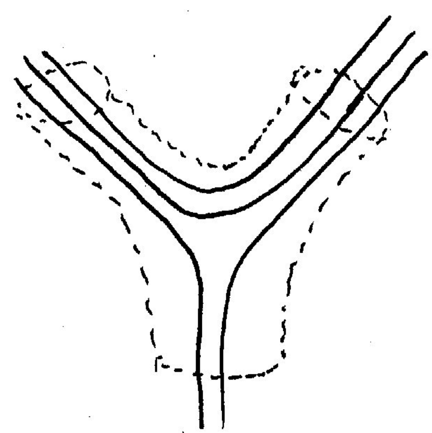
http://torus.math.uiuc.edu/jms/java/dragsphere/

• **Loops** tell us how much a vector rotates due to the curvature of space:

$$\sim U_{(j)}$$

$$U_{(j)} = P\,\text{exp}\left(-i \int_e A_{(j)}\right)$$

• **Spin networks** are linear combinations of $j = 1/2$ loops that are the eigenvectors of geometric operators, e.g. $j_1 = 3/2$, $j_2 = 3/2$, $j_3 = 1$



Attractiveness of loops?

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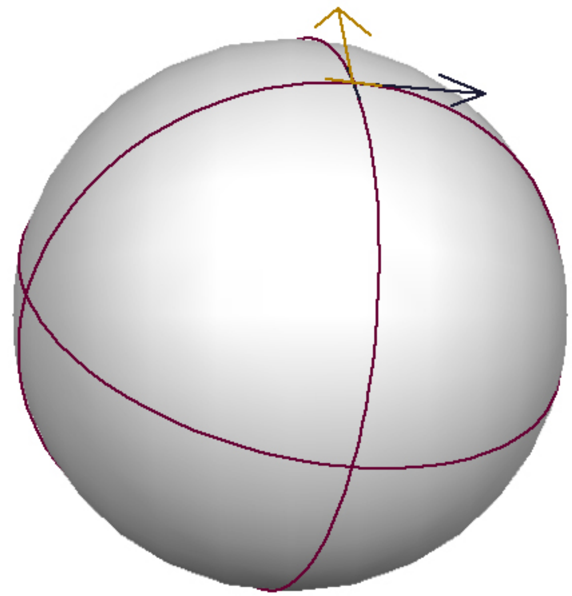

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• Thm: (Baez) **Spin networks** are a basis of the gauge invariant Hilbert space $L^2(\mathcal{A}/\mathcal{G})$.

Adding a cosmological constant

General relativity with new variables (A, E).
[complex $sl(2, \mathbb{C})$-valued connection]

Einstein equations in the (old) new variables:

• **Spatial part** "$G_{ij} = 0$" solutions give possible spatial geometry. ⇒

$$D_a E^a = 0$$
$$E^b F_{ab} = 0$$

• **Temporal part** "$G_{0i} = 0$" tells us how space evolves ⇒

$$E^a \cdot (E^b \times F_{ab} + \frac{\Lambda}{6} \epsilon_{abc} E^b \times E^c) = 0$$

• To recover GR we must implement "reality conditions"

• Seek wavefunctions $\psi(A)$ such that these constraints are satisfied. Can we find a solution? Sure, Kodama did.

Adding a cosmological constant

With the Chern-Simons form

$$S_{CS} = \int_\Sigma (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\, d^3x$$

Let

$$\langle A \mid \psi \rangle = \mathcal{N} \exp\left(-\frac{3}{\Lambda \ell_p^2} S_{CS} \right)$$

$\mathcal{N}$ possibly topology-dependent norm (Soo gr-qc/0109046). The handy fact

$$\frac{\delta}{\delta A_a} S_{CS} = \epsilon^{abc} F_{bc}$$

ensures that the Kodama state satisfies the Hamiltonian constraint.

$$E^a \cdot (E^b \times F_{ab} + \frac{\Lambda}{6} \epsilon_{abc} E^b \times E^c) = 0$$

Also (small) gauge and diffeomorphism invariant.

Adding a cosmological constant

What is this state in the spin network representation? What would be the transform of a spin network state $s(A)$?

$$\begin{aligned}\langle s \mid \psi\rangle &= \sum_{\text{“}A\text{”}} \langle s \mid A\rangle\langle A \mid \psi\rangle \\ &= \int d\mu(A) s(A)\, \psi(A) \\ &= \mathcal{N} \int d\mu(A) s(A) \,\mathrm{exp}\left[\frac{3}{\Lambda \ell_p^2} S_{CS}\right]?\end{aligned}$$

Witten showed that the path integral

$$\Psi(L) = \int d\mu(A) L(A) \,\mathrm{exp}\left[\frac{ik}{4\pi} S_{CS}\right]$$

is, for knots and links L, and real-valued connections equivalent to an invariant, the Kauffman bracket!

Key point: The invariant is sensitive to twists of the spin net edges. PI only defined for framed links - “tubes with stripes” or “ribbons”

Quantum Gravity with cosmological constant?

Beautiful Picture:

- State(s) of Quantum Gravity!
- Includes the cosmological constant!
- Knot classes are label the states!

$$\Psi(s) = K(s)$$

- Has the DeSitter cosmology as a semiclassical limit!
- Cosmological constant - particle statistics connection?

- composite particle statistics determined by framing in theory of fractional QHE

Key new feature: apparently depends on framed spin networks.

But, we do not know what

$$\int d\mu(A)s(A)\ \mathrm{exp}\left[\frac{3}{\Lambda \ell_p^2} S_{CS}\right]$$

means for spin networks $s(A)$ and $sl(2, \mathbb{C})$-valued connections.

Quantum Gravity with cosmological constant?

Obviously this is too good to actually hold.

• Kodama state is in Lorenztian framework. While Witten's result is in YM theory, with real-valued connections

$$K(L) = \int d\mu(A) W(L; A) \exp\left[\frac{ik}{4\pi} S_{CS}(A)\right],$$

does not (obviously) hold for a complex connection. Like defining the inverse Laplace transform due care is required in the choice on contour.

• Is the state normalizable? Not in linearized Lorentzian case [Freidel-Smolin CQG 21 (2004) 3831]

• Violates CPT (relevance? NPT vs. QFT)

• Using the variational calculus methods, the "invariant" for graphs acquires tangent space sensitivity (SM hep-th/9810071)

Quantum Gravity with cosmological constant?

- Due to invariance under large gauge transformations, k is an integer
- Equating YM and Kodama coefficients

$$\frac{ik}{4\pi} = \frac{3i}{\Lambda \ell_p^2} \implies k = \frac{12\pi}{\Lambda \ell_p^2}$$

So $\frac{12\pi}{\Lambda \ell_p^2}$ is an integer. Note: Small Λ means large k.

- The "deformation parameter", a measure of twist, is

$$q = \exp\left(\frac{\pi i}{k+2}\right) \sim \exp\left(\frac{i\,\Lambda\,\ell_p^2}{6}\right)$$

a root of unity.

- Kauffman bracket is a polynomial in q, may be expressed in terms of quantum integers

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$$

and as the evaluation of q spin nets using graphical recoupling theory.

Physics of Quantum Gravity?

A history of an idea (T. Konopka,SAM New J. Phys. **4** (2002) 57)

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Dispersion Relations For massive particles (units with $c = 1$)

$$E^2 = p^2 + m^2$$

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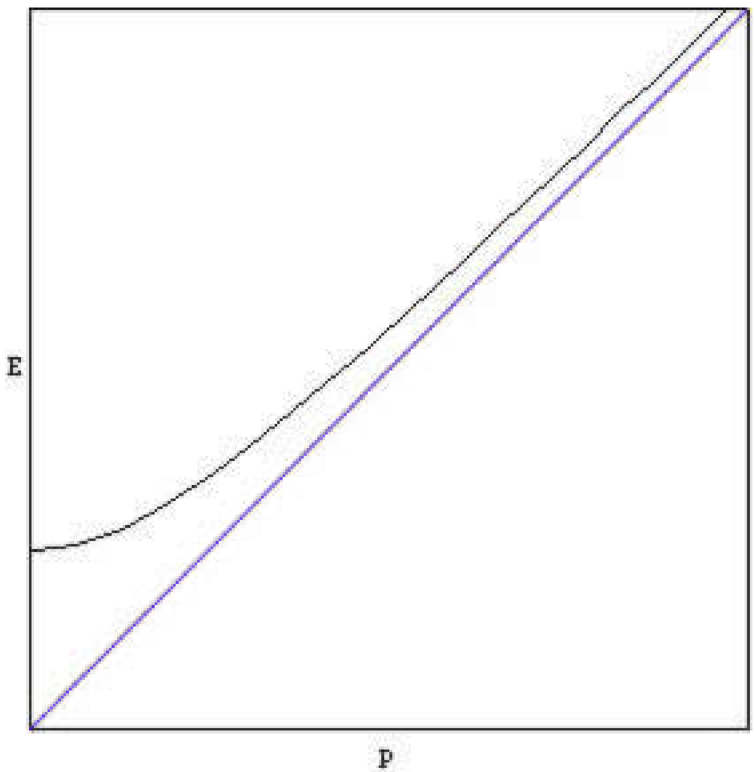
- **Quantum geometry affects the propagation of fields**

Modified Dispersion Relations (MDR) For massive particles (units with $c = 1$)

$$E^2 = p^2 + m^2$$

$$\boxed{E^2 = p^2 + m^2 + \kappa \frac{p^3}{E_P}}$$

Modified Dispersion Relations (MDR)



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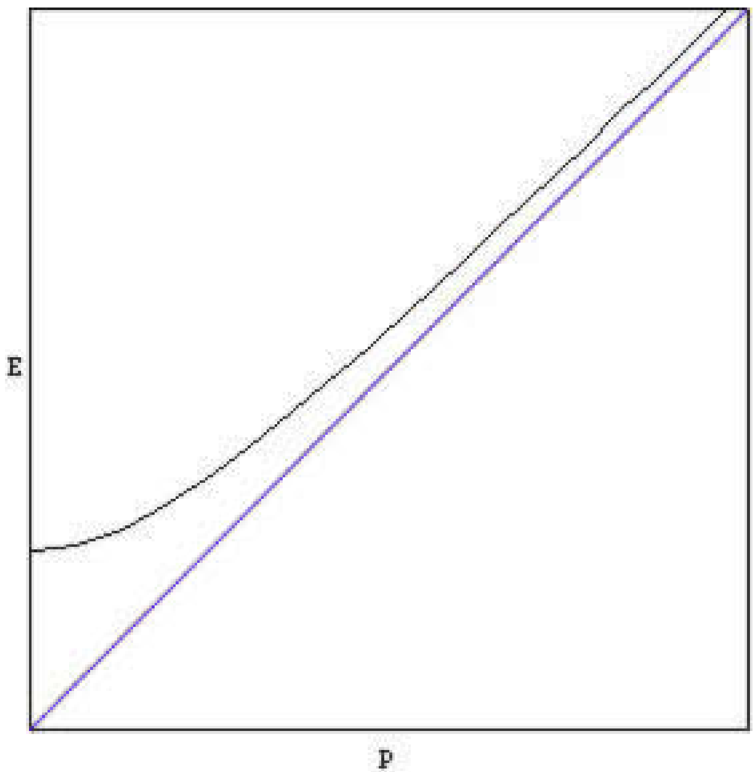
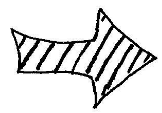
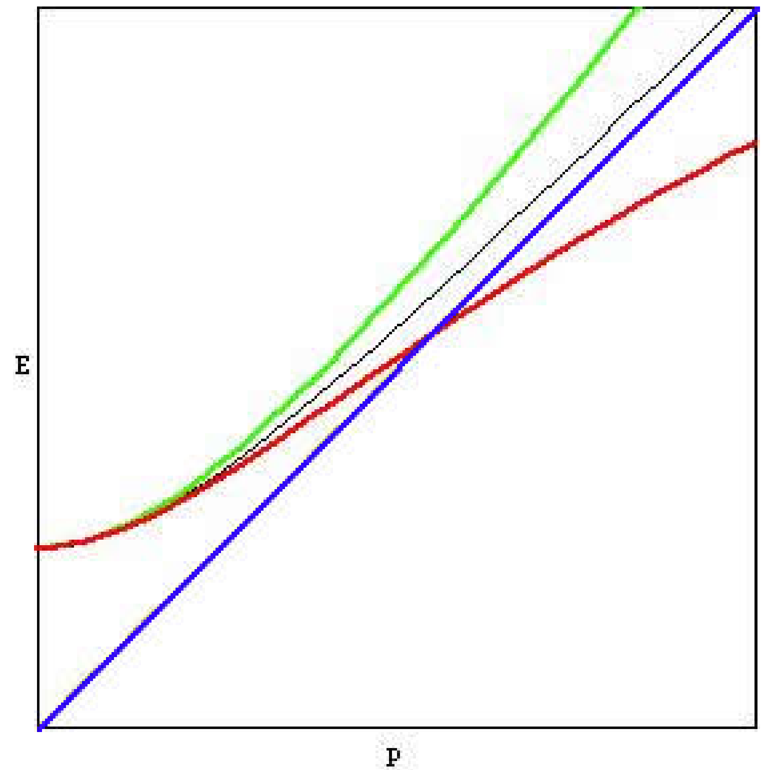






$$E^2 = p^2 + m^2$$

$$E^2 = p^2 + m^2 + \kappa p^3 / E_P$$

Modified Dispersion Relations (MDR)

- κ order unity
- κ is positive or negative
- There is a preferred frame ! Special Relativity is modified!
- Effects are important when $E_{crit} \approx (m^2 E_p)^{1/3} \sim 10^{13}$ and 10^{15} eV for electrons and protons
- Model limited by $p << E_P$

MDR take the leading order form

$$\boxed{E \approx p + \frac{m^2}{2p} + \kappa \frac{p^2}{2E_P}}$$

in which $m << p << E_P$

Model with MDR

- Assume exact energy-momentum conservation
- Assume MDR "cubic corrections"
- Seperate parameters for photons and fermions.

Call photon $\kappa \to \eta$ and electron $\kappa \to \xi$.

Example: **Photon Stability** $\gamma \not\to e^+ + e^-$

SR forbids decay. MDRs allow photon decay. Particle process thresholds are highly sensitive to this kind of modification!

With MDRs the thresholds are

$$p_{\gamma *} = \left[\frac{8 m_e^2 E_P}{(2\eta - \xi)} \right]^{1/3} \quad \text{for } \eta \geq 0$$

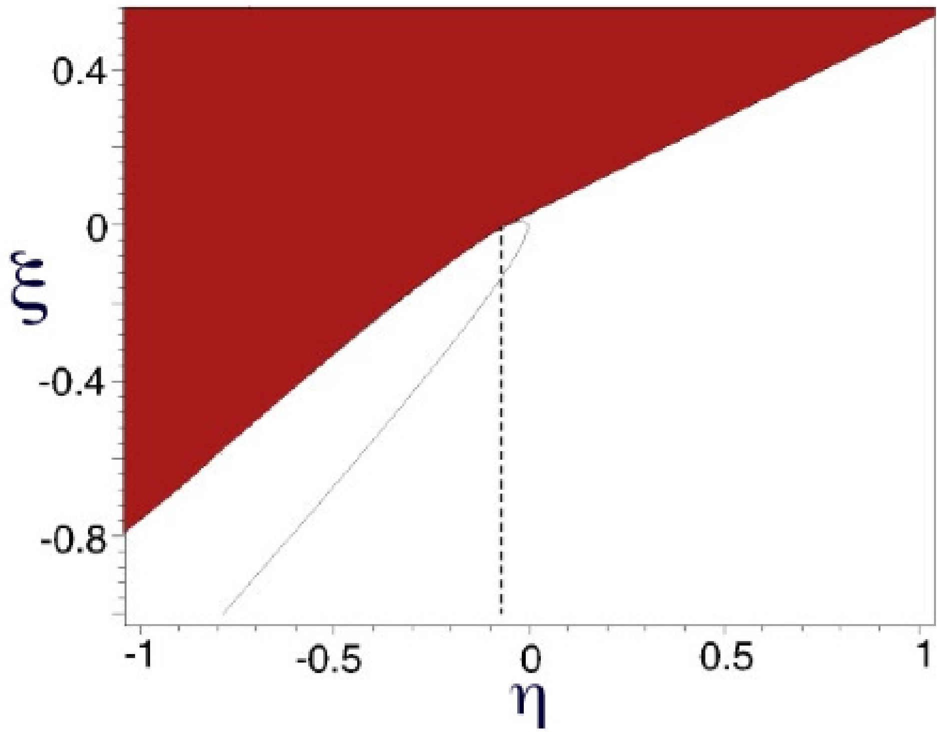
and

$$p_{\gamma *} = \left[\frac{-8 \xi m_e^2 E_P}{(\eta - \xi)^2} \right]^{1/3} \quad \text{for } \xi < \eta < 0$$

Observations of high energy photons produce constraints on ξ and η. e.g. 50 TeV photons from Crab Nebula

A model with MDR

Red region is ruled out by **photon stability** constraint



Planck scale limits!

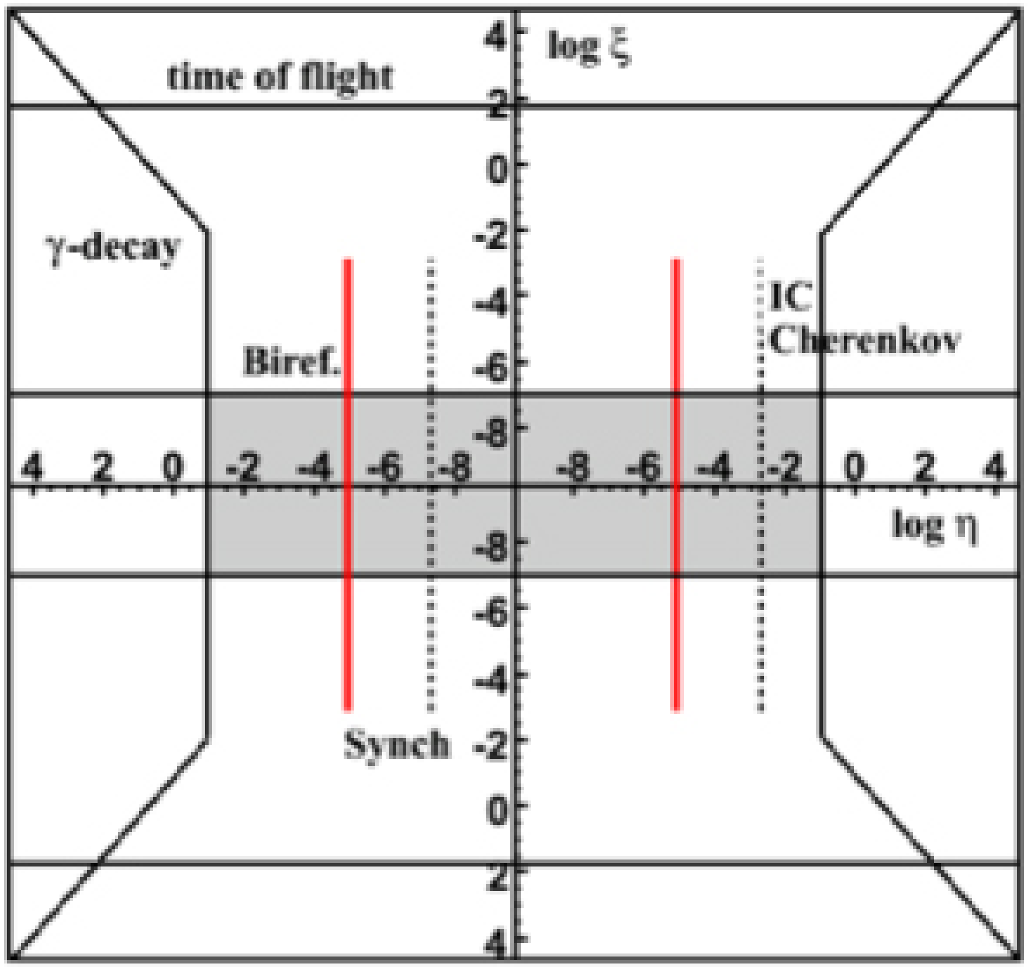
Model with MDR

- Many other processes limit the extent of MDR:

- dispersion
- birefringence
- vaccum Cerenkov radiation
- photon absoprtion
- pion stability
- synchotron radiation
- ultra-high energy cosmic rays
...

Status of MDR 2008

On log scale



Maccinoe, Liberati et. al. arXiv: 0707.2673

Closing question: Are ribbons attractive?

What is the definition of

$$\int d\mu(A) s(A) \exp\left[\frac{3}{\Lambda \ell_p^2} S_{CS}\right]$$

for spin networks $s(A)$ and $sl(2,\mathbb{C})$-valued connections?

For more information

General description of loop quantum gravity:

- Smolin, "Atoms of space and time" Scientific American January 2004
- http://academics.hamilton.edu/physics/smajor/index.html

More technical presentations:

- Baez, http://math.ucr.edu/home/baez/riemannian/
- Thiemann, "Loop Quantum gravity: An inside view" hep-th/0608210
- Smolin "Quantum Gravity with a positive cosmological constant" hep-th/0209079
- Rovelli, *Quantum Gravity* (Cambridge, 2004)
- Thiemann *Modern Canonical Quantum General Relativity* (Cambridge, 2007)
- Major, Smolin "Quantum Deformation of Quantum Gravity" Nuc. Phys. B 473 (1996) 267 gr-qc/9512020
- Major, "On the q-quantum gravity loop algebra"