Reference Frames
HW #1, Due 9/3/04

Read T&R Ch.1 (read quickly), 2.1-2.4

1) Galeean Transformation. A train moving with a speed of 50 m/s passes through a railway station at time t=t'=0. Fifteen seconds later a firecracker explodes on the track 1.0 km away from the train station in the direction that the train is moving. Find the coordinates (position and time) of this event in both the station frame (Home Frame) and the train frame (Other Frame).

2) Briefly describe an inertial reference frame. Since it takes time for light to travel from an event to the origin, an observer at the origin “sees” an event a short time after the event happens. Describe one way that our reference frame can take this travel time into account. (This travel time is a nuisance, but it does not cause the problems with simultaneity that we will soon meet.)

3) Use Excel to graph \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \) for 0<\(\beta<0.995\). Since \( \beta = v/c \), you are really plotting \( \gamma \) as velocity \( v \) approaches the speed of light. We will spend quite a bit of time working with \( \gamma \) and \( \beta \) in the next few weeks. Make sure that you use enough points to show the shape of the curve.

4) Use a Taylor expansion to show that \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} ≈ 1 + \frac{1}{2} \beta^2 \) for \( v << c \) (small velocities as compared to the speed of light). You can make your life easier by considering \( \beta^2 \) as one entity.

5) The Michelson-Morley experiment: Assume that interferometer arm #1 is in the +x direction and interferometer arm #2 is in the +y direction. Each arm has length L in the lab frame. Further assume that the whole experiment is moving in the +x direction with respect to the ‘ether’ (snicker, snicker). Assume that light travels at \( c = 3e8 \) m/s with respect to the ‘ether.’

   a) Find the velocity in the lab frame (speed along the ground) of the light in
      i) arm 2 (perpendicular to motion)
      ii) arm 1, first leg (in the direction of the motion of the lab, against the “flow” of ether)
      iii) arm 1, second leg (along with the “flow” of ether)

   b) Show that the total time traveled on arm 2 is given by \( T_2 = \frac{2L}{c} \frac{1}{\sqrt{1 - \beta^2}} \) and the total time in arm 1 is given by \( T_1 = \frac{2L}{c} \frac{1}{1 - \beta^2} \).

   c) Use Taylor expansions to show that \( \Delta T = T_1 - T_2 = L\beta^2 \). Note the following Taylor expansions: \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} ≈ 1 + \frac{1}{2} \beta^2 \) and \( \gamma^2 = \frac{1}{1 - \beta^2} ≈ 1 + \beta^2 \).

   d) Extra Credit: Imagine watching this experiment from the “ether” frame. Show that if you contract the length of arm 1 (in the direction of the motion) by \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} ≈ 1 + \frac{1}{2} \beta^2 \) you get no time difference. Lorentz made this argument 15 years before Einstein’s theory of relativity.

6) T&R problem 2.6
1) Remember a frame of reference has a clock at every point so you do not need to take the speed of sound into account.

Station Frame: \[ t = 15 \text{ s} \quad x = 1000 \text{ m} \]
Train Frame: \[ d = vt \]

In 15 seconds, the train traveled: \[ vt - 50 \% \times 15 \text{ s} = 750 \text{ m} \]

\[ t' = 15 \text{ s} \quad x' = 1000 \text{ m} - 750 \text{ m} = 250 \text{ m} \]

OR \[ x' = x - vt = 1000 \text{ m} - 50 \% \times 15 \text{ s} = 250 \text{ m} \]

2) A reference frame is a way to measure the coordinates \((t, x, y, z)\) of events. It is a set of axes through all space measured with respect to one origin and one clock.

An inertial reference frame is a reference frame that is not accelerating. Newton's laws (and other laws of physics) will work in a frame that has a constant velocity. For instance, an object at rest will tend to remain at rest.

In order to measure the coordinates of any event, a reference frame may consist of clocks placed at a lattice throughout space. This way there is always a clock near any event and there is no need to account for information travel time.
2) The observer may simply measure the distance to each event and calculate the light travel time between the event and "seeing" the event at the origin. If a flash of light is sent from the origin in time to reflect off the event back to the origin, then the total light travel time gives the distance to the event \( d = \frac{1}{c} \frac{c}{t} \) and the event must have happened midway between sending and receiving the light \( t = \frac{1}{2} (c^2 + t) \).

3) See attached.

4) Let \( X = \beta^2 \)

\[
Y = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - X}} = \left(1 - X\right)^{-\frac{1}{2}}
\]

\[
f(x_0 + x) \approx f(x_0) + x f'(x_0) + \frac{x^2}{2} f''(x_0) + \ldots
\]

\[
f(x) \approx f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \ldots
\]

\[
f(x) = \left( 1 - x \right)^{-\frac{1}{2}} \quad f(0) = 1
\]

\[
f'(x) = -\frac{1}{2} (1 - x)^{-\frac{3}{2}} \quad f'(0) = \frac{1}{2}
\]

\[
f''(x) = -\frac{1}{2} (1 - x)^{-\frac{5}{2}} \quad f''(0) = \frac{1}{2}
\]

\[
f(x) \approx 1 + x \frac{1}{2} + \ldots
\]

\[
f'(\beta^2) = 1 + \frac{1}{2} \beta^2 + \ldots
\]

\[
Y = \frac{1}{\sqrt{1 - \beta^2}} \approx 1 + \frac{1}{2} \beta^2 + \ldots
\]
\( c = \text{Speed of Light Through the Ether} \)

5) \( \text{a) } V_e \rightarrow \)

\( V_e = \text{Speed of Apparatus through the Ether} \)

\[\text{Consider only the first part} \quad \frac{c}{V_y} \]

\[V_y = \text{Speed through the Ether to keep up with apparatus} \]

\[c = \text{Speed of Light through the Ether} \]

\[V_y = \text{Velocity in the Lab Frame} \]

\[c^2 = V_y^2 + V_e^2 \]

\[V_y = \sqrt{c^2 - V_e^2} \]

\[\text{arm 1, First Leg: against the "Flow of the Ether"} \]

\[V = c - V_e \]

\[\text{arm 2, Second Leg: with the Flow of the Ether} \]

\[V = c + V_e \]

Note: \( \beta = \frac{V_e}{c} \)

b) Arm 2: \( d = \sqrt{t} \)

\[t_e = \frac{d}{V_e} = \frac{2L}{\sqrt{c^2 - V_e^2}} = \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - \left(\frac{V_e}{c}\right)^2}} = \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - \beta^2}} \]

Arm 1: \( t_1 = t_{\text{Leg 1}} + t_{\text{Leg 2}} = \frac{L}{c - V_x} + \frac{L}{c + V_x} \)

\[= \frac{L}{c - V_x} \left( \frac{c + V_x}{c + V_x} \right) + \frac{L}{c + V_x} \left( \frac{c - V_x}{c - V_x} \right) \]

\[= \frac{LC + LV_x + LC - LV_x}{c^2 - V_x^2} = \frac{2Lc}{c^2 - V_x^2} = \frac{2L}{c} \cdot \frac{1}{1 - \left(\frac{V_x}{c}\right)^2} \]
\[ \beta = \frac{V_0}{c} \]

\[ t_2 = \frac{2L}{c} \sqrt{\frac{1}{1-\beta^2}} \]
but \[ \sqrt{\frac{1}{1-\beta^2}} \approx 1 + \frac{1}{2} \beta^2 \]

\[ t_2 \approx \frac{2L}{c} \left( 1 + \frac{1}{2} \beta^2 \right) \]

Arm 1: \[ t_1 = \frac{2L}{c} \frac{1}{1-\beta^2} \]

Taylor Expand \[ \frac{1}{1-\beta^2} \]

Let \[ x = \beta^2 \]

\[ f(x) = \frac{1}{1-x} = (1-x)^{-1} \quad f(0) = 1 \]

\[ f'(x) = -1 \cdot (1-x)^{-2} = (-x)^2 \quad f'(0) = 1 \]

\[ f(x) \approx 1 + x + x^2 + \ldots \]

\[ \frac{1}{1-x} \approx 1 + x \]

\[ \frac{1}{1-\beta^2} \approx 1 + \beta^2 + \ldots \]

\[ t_1 = \frac{2L}{c} \frac{1}{1-\beta^2} \approx \frac{2L}{c} \left( 1 + \beta^2 \right) \]

\[ \Delta T = t_2 - t_1 = \frac{2L}{c} \left( 1 + \frac{1}{2} \beta^2 \right) - \frac{2L}{c} \left( 1 + \beta^2 \right) \]

\[ = \frac{2L}{c} \left( \frac{1}{2} \beta^2 - \beta^2 \right) = -\frac{2L}{c} \frac{1}{2} \beta^2 = -\frac{L}{c} \beta^2 \]

Note: When we rotate the apparatus by 90°
We get the same time difference in the opposite direction. The total time shift is
\[ 2 \Delta T = 2 \frac{L}{c} \beta^2 \left( -\frac{V_0^2}{c^2} \left( 1 + 1/2 \right) \right) \]

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Extra Credit

\[ T_1 > T_2 \]

If we let \( L_1 \to L_1/\gamma \) then the time \( T_1 \) will also get smaller by a factor \( \gamma \).

\[ T_1 \to T_1/\gamma = T_1 \sqrt{1-\beta^2} = T_1 \sqrt{1-\beta^2} \]

\[ T_{\text{New}} = \frac{2L}{c} \frac{1}{\sqrt{1-\beta^2}} \cdot \sqrt{1-\beta^2} = \frac{2L}{c} \frac{1}{\sqrt{1-\beta^2}} \]

\[ = T_2 \]

So \( \Delta T = T_1 - T_2 = 0 \) This would account for the Michelson Morley result.

Note: You can simply use \( \frac{T_1}{T_2} = \sqrt{1-\beta^2} = \gamma \) to find the same result.
\[ \Delta T = \frac{v^2}{c^2} \left( \frac{1}{l_1} + \frac{1}{l_2} \right) \]

If the interference pattern shifts by 1 fringe, the correspond to one period,

\[ \lambda = 589 \text{ nm} \]
\[ \frac{c}{\lambda} = \frac{c}{589 \times 10^{-9} \text{ m}} = \frac{3 \times 10^8 \text{ m/s}}{589 \times 10^{-9} \text{ m}} = 5 \times 10^{11} \frac{\text{s}}{\text{m}} \]

\[ \tau = \frac{1}{f} = \frac{1}{5 \times 10^{14} \text{ Hz}} = 2 \times 10^{-15} \text{ sec} \]

0.005 fringes correspond to a time of

\[ \Delta T = 0.005 \times 2 \times 10^{-15} \text{ sec} = 1 \times 10^{-17} \text{ sec} \]

\[ 1 \times 10^{-17} \text{ sec} > \frac{v^2}{c^2} \left( \frac{1}{l_1} + \frac{1}{l_2} \right) = \]

\[ \frac{v^2}{c^2} < 1 \times 10^{-17} \text{ sec} \cdot \frac{3 \times 10^8 \text{ m/s}}{2 \times 11 \text{ m}} = 1.4 \times 10^{-10} \]

\[ \frac{v}{c} < \sqrt{1.4 \times 10^{-10}} = 1.2 \times 10^{-5} \]

\[ v < 1.2 \times 10^{-5} c = 3.52 \times 10^7 \text{ m/s} \]

Note: The Earth is traveling around the Sun at 30,000 m/s
\[ \beta = 0 + 0.05 \]
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]

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**HW#1 Problem 3**

- **gamma**
- **beta (v/c)**

Values range from 0.998 to 15.81929993.