Note: The most important part of each question is Setting Up The Problem, or describing what you should do. Don't get frustrated over Difficult Derivatives or Not being able to find the appropriate integral.

Given Enough Time (in a future life perhaps) I am confident that you can get the algebra correct.

Note: 8. pay a lot of attention to the Method.

9. Skip This For Now

10. Pay Special attention to Setting up the Integral

Note: 6. Should Not take a lot of time, but require a lot of Understanding
1) \( y(x) = A \sin(\pi x) \) for \( 0 < x < 1 \)

**Normalize \( y \) (find \( A \))**

\[
1 = \int_0^1 y^2 dx = \int_0^1 4x^2 dx
\]

\[
= \int_0^1 4 \sin^2(\pi x) \, dx = A^2 \int_0^1 \sin^2(\pi x) \, dx
\]

\[
= A^2 \cdot \frac{3}{2}
\]

\[
A = \sqrt{\frac{2}{3}}
\]

**Prob. (\( 0 < x < \frac{1}{2} \))**

\[
= \frac{1}{4} \int_0^1 y^2 dx = \frac{1}{4} \int_0^{\frac{1}{4}} \sin^2(\pi x) \, dx
\]

\[
= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}
\]

**Note:** The average value of \( \sin^2 \theta \) or \( \cos^2 \theta \) over one period is \( \frac{1}{2} \).

2) See 3a. in your notes.

**For Infinite Square Well**

\[
E = \frac{n^2 \hbar^2}{2mL^2}
\]

\( n = 1 \) in an atom \( m = 514 K eV c \)

\[
L \approx 2A = 92 \text{ a.u.}
\]

\[
E = \frac{\pi^2 \frac{\hbar^2 c^2}{2m} n^2}{2 \cdot 5.1 \times 10^{-21} \text{eV} (2A)^2} = 10^{-6} \text{eV} n^2
\]

\( E_1 = 10 \text{eV} \) \( E_2 = 40 \text{eV} \) \( E_3 = 90 \text{eV} \)

**Atomic Energy Scale**

\( m \) in nucleus \( m = 938 M eV c^2 \)

\[
L \approx 2 \text{ a.m } = 10^{-15} \text{m} = 10^{-9} \text{m}
\]

\[
E = \frac{\pi^2 \frac{\hbar^2 c^2}{2m} n^2}{2 \cdot 9.38 \times 10^{-21} \text{eV} (10^{-9} \text{m})^2} = 210 \text{ MeV}
\]

\( E_1 = 50 \text{ MeV} \) \( E_2 = 200 \text{ MeV} \) \( E_3 = 450 \text{ MeV} \)

A little high for nuclear scale.
You would not need to do this part.

\[ A = \frac{\pi^2}{2} \]

Normalized:

\[ A = \frac{1}{\pi^2} \]

\[ E = \frac{2A\pi^2}{x^4} \]

\[ y(0) = 0 \]

\[ y(x) = A \sin \left( \frac{\pi x}{L} \right) \]

Solution works if this integral exists.

\[ y = \sin \left( \frac{\pi x}{L} \right) \]

\[ k = \frac{n \pi}{L} \]

Try this solution:

\[ y(x) = \sin \left( \frac{n \pi x}{L} \right) \]

\[ y = 0 \] at the top.

At the bottom,

\[ y = 0 \]

So,

\[ y(x) = A \sin \left( \frac{\pi x}{L} \right) \]

Guess:

\[ y(x) = c \sin \left( \frac{\pi x}{L} \right) \]

\[ y(x) = c \sin \left( \frac{\pi x}{L} \right) \]

Satisfy this:

Solution works if this integral exists.
3a) \( y(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n \pi x}{L} \right) \)

3b) \( n = 2 \)

\[ \langle X \rangle = \int_{-a}^{a} x \, 4 \, dx = x \]

\[ = \frac{1}{2} \int_{-a}^{a} 4 \, dx = \frac{1}{2} 2a = a \]

\[ \int_{-a}^{a} \sin^2 \left( \frac{n \pi x}{L} \right) dx = \frac{a^2}{2} \]

\[ \int_{-a}^{a} \cos^2 \left( \frac{n \pi x}{L} \right) dx = \frac{a^2}{2} \]

\[ \frac{17}{17} \frac{10}{10} \]

\[ \int_0^{2a} \sin^2 (2x) \, dx = \frac{a^2}{2} \]

\[ = \frac{2}{L} \left( \frac{a^2}{4} - 0 - 0 \right) - (0 - 0 - 0) \]

\[ = \frac{a^2}{L} \]

3c) \( n = 2 \)

\[ \langle P \rangle = \int_{-a}^{a} x \, p \, 4 \, dx \]

\[ = \frac{1}{2} \int_{-a}^{a} 4 \, dx = \frac{1}{2} 2a = a \]

\[ \int_{-a}^{a} \sin \left( \frac{n \pi x}{L} \right) \, dx = 0 \]

\[ \int_{-a}^{a} \cos \left( \frac{n \pi x}{L} \right) \, dx = 0 \]

\[ \int_{-a}^{a} \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi x}{L} \right) \, dx = \frac{a^2}{2a} \]

3d) \( n = 2 \)

\[ \langle P^2 \rangle = \int_{-a}^{a} x^2 \, p^2 \, 4 \, dx \]

\[ = \frac{1}{2} \int_{-a}^{a} 4 \, dx = \frac{1}{2} 2a = a \]

\[ \int_{-a}^{a} \sin^3 \left( \frac{n \pi x}{L} \right) dx = \frac{a^2}{2} \]

\[ \int_{-a}^{a} \cos^3 \left( \frac{n \pi x}{L} \right) dx = \frac{a^2}{2} \]

\[ \frac{17}{17} \frac{19}{19} \]

\[ \int_{-a}^{a} \sin^3 \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) dx = \frac{a^2}{2a} \]

\[ = \frac{2}{L} \left( \frac{a^2}{4} - 0 - 0 \right) - (0 - 0 - 0) \]

\[ = \frac{a^2}{L} \]
\[
\langle p^2 \rangle = -\frac{2\hbar^2}{\ell^2} \int \sin \left( \frac{\pi x}{\ell} \right) \left( -\frac{\hbar^2}{\ell^2} \right) \sin \left( \frac{\pi x}{\ell} \right) dx
\]
\[
= \frac{2\hbar^2 \pi^2}{\ell^2} \int_0^\ell \sin^2 \left( \frac{\pi x}{\ell} \right) dx
\]
\[
= \frac{2\hbar^2 \pi^2}{\ell^2} \cdot \frac{\ell}{2} = \frac{\pi^2 \hbar^2}{\ell^2}
\]

See Normalization in 3a

\[
\langle p^2 \rangle = \frac{\pi^2 \hbar^2}{\ell^2}
\]

Note: \( \langle p^2 \rangle \neq \langle p \rangle^2 = 0^2 = 0 \)

The average momentum \( \langle p \rangle \) is 0 because the particle spends as much time going left as going right.

The average momentum squared \( \langle p^2 \rangle \) is not zero because \( p^2 \) is always positive if the particle is moving left or right.

Consider 4 and \( 4' = e^{i\theta}4 \)

\[
\langle 0 \rangle = \int 4*\delta 4'dx = \int (e^{i\theta}4) * \delta (e^{i\theta}4) dx = \int e^{-i\theta}4* \delta e^{i\theta}4 dx = \int e^{-i\theta} e^{i\theta} 4* \delta 4 dx = \int (\delta(\theta-\theta)) 4* \delta 4 dx = \int 4* \delta 4 dx
\]

You can multiply a wave function by 1, -1, i, -i or any other complex number (of magnitude 1) without changing the physical meaning.
5) 
False

For potential wells, there are only valid solutions to the Schrödinger Equation at very specific energies. This is what is meant by quantized energies.

6) Bond States

Note:
- Curve near where E-V is big
  - Short, A small amplitude
- Extend into forbidden region
- Exponential tail goes toward O as x→∞
  - but always curve away from O

AND/OR

Note:
- Short, 7 where E-V is big
- Always curve away from 0 when E < V

Unbond State: 2 Shots where E-V is big.
7) Finite Square Well

![Graph of a finite square well]

\[ \text{Large } \lambda \]

Infinity Square Well

\[ \text{Short } \lambda \]

\[ \gamma = \frac{1}{\hbar^2} \text{ Large } \hbar \]

\[ \epsilon = \frac{\epsilon^2}{2m} \text{ Large } \epsilon \]

Higher Walls Shift the Energy Levels Up

8) Simple Harmonic Oscillator

\[ V(x) = \frac{1}{2} m \omega^2 x^2 \]

\[ W = \sqrt{\frac{k}{m}} \quad k = \text{Spring Const.} \]

Schrödinger's Equation is (Time Independent)

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + V(x) \psi = \epsilon \psi \]

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + \frac{1}{2} m \omega^2 x^2 \psi = \epsilon \psi \]

\[ q) \quad \psi(x) = Ae^{-ax} \]

\[ -\frac{\hbar^2}{2m} \frac{d^3}{dx^3} A e^{-ax} + \frac{1}{2} m \omega^2 x^2 A e^{-ax} = \epsilon A e^{-ax} \]

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} A e^{-ax} + \frac{1}{2} m \omega^2 x^2 A e^{-ax} = \epsilon A e^{-ax} \]

\[ -\frac{\hbar^2}{2m} a^2 + \frac{1}{2} m \omega^2 x^2 = \epsilon \]

In order to be a Solution, This must be true at all Values of \( x \):

Consider \( x = 0 \)

\[ -\frac{\hbar^2}{2m} a^2 \neq \epsilon \]

This defines \( \epsilon \) in terms of Constants.

For other \( x \) we have

\[ \frac{1}{2} m \omega^2 x^2 + \frac{\hbar^2 a^2}{2m} = 0 \]

\[ x = 0 \]

4. Only Work at One Value of \( x \)

\textit{NOT A VALID SOLUTION}
Try $y = A e^{-ax^2}$

\[-\frac{h^2}{2m} \frac{d^2}{d x^2} y + \frac{1}{2} m w^2 x^2 y = E y\]

\[-\frac{h^2}{2m} \frac{d^2}{d x^2} A e^{-ax^2} + \frac{1}{2} m w^2 A e^{-ax^2} = E A e^{-ax^2}\]

Note: \( \frac{d}{dx} A e^{-ax^2} = A e^{-ax^2} (-2ax) \)

\[\frac{d^2}{dx^2} A e^{-ax^2} = \frac{d}{dx} A (-2ax) e^{-ax^2}\]

\[= -2Aax \left( e^{-ax^2} + x e^{-ax^2} (-2ax) \right)\]

\[= -2A A e^{-ax^2} + 4a^2 x^2 e^{-ax^2}\]

Note: \((fg)' = f'g + fg'\)

\[\left( -\frac{h^2}{2m} \left( -2A A e^{-ax^2} + 4a^2 x^2 e^{-ax^2} \right) + \frac{1}{2} m w^2 A e^{-ax^2} \right) = E A e^{-ax^2}\]

\[\left( \frac{2a h^2}{2m} - \frac{4a h^2}{2m} x^2 + \frac{1}{2} m w^2 x^2 \right) = E\]

This must work for all \(x\); this will only be true if the \(x^2\) terms cancel and the constant term cancel.

Consider \(x = 0\):

\[\frac{2a h^2}{2m} = E\]

If this is true then (plug back in)

\[-\frac{4a h^2}{2m} x^2 + \frac{1}{2} m w^2 x^2 = E - \frac{2a h^2}{m} \]

Only \(a\) is unknown

\[a = \frac{m w}{2h}\]

\[E = \frac{2a h^2}{m} = \frac{m w}{2h} \frac{h^2}{m}\]

So \(E = \frac{1}{2} \hbar w\)

\[y(x) = A e^{-ax^2}\] Works IFF \(a = \frac{m w}{2h}\)

and \(E = \frac{1}{2} \hbar w\)
8d) **Normalization**:

\[ Y(x) = A e^{-ax^2} = A e^{-\frac{mw}{2\hbar} x^2} \]

\[ a = \frac{m}{2\hbar} \]

\[ 1 = \int_{-\infty}^{\infty} Y(x) dx \]

\[ = \int_{-\infty}^{\infty} A e^{-ax^2} dx \]

\[ = A^2 \int_{-\infty}^{\infty} e^{-x^2} dx \]

\[ = 2A^2 \int_{0}^{\infty} e^{-x^2} dx \]

Note: \( Y \) is symmetric about \( x=0 \)

\[ \int_{0}^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{b}} \]

\[ b = 2a \]

\[ A = \sqrt{\frac{\pi}{2a}} = \frac{\sqrt{\pi}}{\sqrt{2a}} \]

\[ A = \sqrt{\frac{\pi}{2a}} \]

\[ \alpha = \frac{m}{2\hbar} \]

\[ A = \sqrt{\frac{m}{\pi \hbar}} \]

\[ Y(x) = A e^{-ax^2} \]

\[ \Phi(x,t) = Y(x) e^{-i\frac{\varepsilon}{\hbar} t} \]

\[ \Phi(x,t) = \sqrt{\frac{m}{\pi \hbar}} e^{-\frac{mw}{2\hbar} x^2} e^{-i\frac{\varepsilon}{\hbar} t} \]

\[ \varepsilon = \frac{1}{2} \hbar \omega \] For this bound state.

\[ \text{When!} \]

\[ p290 +hw9 \]

\[ p8 \]
9) Skip This Problem For Now

10) a) \( \sum_{n=1}^{\infty} \text{odd } n \ dx = 0 \) 

b) \( \langle \psi \rangle = \int_{-\infty}^{\infty} \psi^* \psi \ dx = \int_{-\infty}^{\infty} \psi^* \psi \ dx \)

\[ \int_{-\infty}^{\infty} \psi^* \psi \ dx = \int_{-\infty}^{\infty} x^2 \ e^{-2ax^2} \ dx \]

Even function

So \( x \ e^{-2ax^2} \) is odd

\[ x \rightarrow -x \]

\[ -x \ e^{-2a(-x)^2} = -1 \cdot x \ e^{-2ax^2} \]

\[ = A^2 \int_{-\infty}^{\infty} \text{odd function} \ dx \]

\[ = 0 \]

If you look at a pendulum, it averages zero position at \( x = 0 \)

c) \( \langle p \rangle = \int_{-\infty}^{\infty} p \ \psi^* \psi \ dx = \int_{-\infty}^{\infty} p \ \psi^* \psi \ dx \)

\[ = A \int_{-\infty}^{\infty} x \ e^{-2ax^2} \ dx \]

\[ = A \int_{-\infty}^{\infty} x \ e^{-2ax^2} \ dx \]

\[ = A^2 \int_{-\infty}^{\infty} x^2 \ e^{-2ax^2} \ dx \]

See part b

\[ = 0 \]

d) A pendulum spends equal amounts of time on the left and on the right. So \( \langle x \rangle = 0 \)

A pendulum spends equal amounts of time traveling to the left and to the right. So \( \langle p \rangle = 0 \)

e) \( \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \ \psi^* \psi \ dx = \int_{-\infty}^{\infty} x^2 \ e^{-2ax^2} \ dx \)

\[ = A^2 \int_{0}^{\infty} x^2 \ e^{-2ax^2} \ dx \]

\[ \int_{0}^{\infty} x^2 \ e^{-2ax^2} \ dx = \frac{18.77}{(\sqrt{2)})^3} = \frac{18.77}{2^{3/2}} = 2.63 \]

\[ = A^2 \int_{0}^{\infty} x^2 \ e^{-2ax^2} \ dx \]

\[ = \frac{\sqrt{\pi}}{4} \left( \frac{1}{2a} \right)^{3/2} = \frac{m \mu}{4} \left( \frac{1}{2a} \right)^{3/2} = \frac{m \mu}{4} \left( \frac{1}{2 \cdot \frac{\hbar}{2m}} \right)^{3/2} \]

\[ = \frac{m \mu}{4} \left( \frac{1}{2 \cdot \frac{\hbar}{2m}} \right)^{3/2} = \frac{\sqrt{\pi}}{2 \cdot \frac{\hbar}{2m}} = \frac{\sqrt{\pi}}{\frac{\hbar}{2m}} \]

\[ \frac{1}{\sqrt{2 \cdot \frac{\hbar}{2m}}} = \frac{1}{\sqrt{2 \cdot 2m \hbar}} = \frac{1}{2 \cdot \frac{\hbar}{2m}} \]