Thickness-Two Graphs Part One: New Nine-Critical Graphs, Permuted Layer Graphs, and Catlin's Graphs

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Abstract

The purpose of this paper is to offer new insight and tools toward the pursuit of the largest chromatic number in the class of thickness-two graphs. At present, the highest chromatic number known for a thickness-two graph is 9, and there is only one known color-critical such graph. We introduce 40 small 9-critical thickness-two graphs, and then use a new construction, the *permuted layer graphs*, together with a construction of Hajós to create an infinite family of 9-critical thickness-two graphs. Finally, a non-trivial infinite subfamily of Catlin's graphs, with directly computable chromatic numbers, is shown to have thickness two.

1 Introduction

Decomposing a graph into planar layers is of primary importance in the field of VLSI (see, for example, [GJS76]). To this end, the *thickness* of a graph G, denoted $\Theta(G)$, is the smallest integer t so that G can be represented as the union of t planar graphs. We then say that G is *thickness* t [Wes01]. Thus planar graphs have thickness one, while K_5 and $K_{3,3}$ have thickness two. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G can be properly vertex colored with k colors. We then say that G is k-*chromatic*. Thus, for example, if G is planar then $\chi(G) \leq 4$ [AH76, RSST96]. A graph is said to be k-*critical* if it is k-chromatic but every proper subgraph can be properly colored with fewer than k colors.

Given an arbitrary graph G and fixed positive integer $t > 1$, verifying that $\Theta(G) = t$ is an NP-hard problem [Man83]. Similarly, given an arbitrary graph G and fixed positive integer $k > 2$, verifying that $\chi(G) = k$ is also NP-hard [GJ90, GJS74]. Combining these two ideas leads to a longstanding and difficult open problem [Rin59]: "What is the largest chromatic number that occurs in the class of thickness-two graphs?"

It is well-known that the largest chromatic number of a thickness-two graph lies between 9 and 12 inclusive [JT95, Hut93]. That is, if G is an arbitrary thickness-two graph then $\chi(G) \leq N$, where

$$N \in \{9, 10, 11, 12\}. \tag{1}$$

The largest element in (1) comes from a straightforward induction argument that relies on Euler's Formula for plane graphs. The smallest element in (1) is due to the existence of the 9-critical thickness-two graph given by $S = K_6 \vee C_5$, which is called *Sulanke's graph*. The third author proved that S had thickness two in 1973 and the result was reported in 1980 in [Gar80]. Since then no progress has been made toward closing this gap of possibilities. Further, no other examples of 9-critical thickness-two graphs have emerged. The desideratum for the main question would be to find an asymptotically sharp bound for N in (1). Short of that difficult goal, producing a 10-chromatic thickness-two graph would be of paramount importance. In order to construct the latter, new techniques are needed, as well as a large collection of 9-critical thickness-two graphs. It is the latter upon which we focus our attention in this paper and the sequel [GS].

In particular, the purpose of this paper is threefold. First a new class of thickness-two graphs, called *permuted layer graphs*, is introduced and shown to contain all thickness-two graphs as subgraphs. Thus we can study permuted layer graphs to obtain the chromatic properties of thickness-two graphs. Second, we introduce infinitely many 9-critical thickness-two graphs and we do so in two stages. In particular we give a list of 40 new 9-critical thickness-two graphs on 12 through 16 vertices. Moreover, to add further ballast to the lower bound of 9 in (1), we use a construction of Hajós together with the Permuted Layer construction to generate infinitely many 9-critical thickness-two graphs. To strengthen the latter, we characterize precisely when the construction of Hajós produces a graph that has thickness t. Third, an infinite subfamily of Catlin's graphs [Cat79] with easily computable chromatic numbers is shown to have thickness two. We end with a list of open questions.

2 Permuted Layer Graphs

In this section we define a restricted type of thickness-two graph, called a permuted layer graph, and identify two especially nice properties of the smaller class. First, permuted layer graphs are easy to construct. Second, every maximal thickness-two graph is a permuted layer graph. Thus there is a thickness-two graph with chromatic number as large as k if and only if there is a permuted layer graph with chromatic number as large as k. The following construction defines the new subclass of graphs.

The Permuted Layer Construction. Let H be a planar graph with $V(H) = \{v_1, \ldots, v_n\}$, and let σ be a permutation of $V(H)$. Construct another planar graph H' isomorphic to H with vertices labeled so that the vertex corresponding to v_i in H is labeled $\sigma(v_i)$ in H'. Identify H and H' at vertices of the same label and call the resulting graph $\tilde{G}$; that is $\tilde{G} = H \cup H'$. Since $\tilde{G}$ may have multiple edges, we let G be the underlying simple graph and call it a *permuted layer graph with base graph* H. In the special situation when $\tilde{G}$ does not have multiple edges we call G $(= \tilde{G})$ a *full permuted layer graph.*

A more direct, but slightly less intuitive, way to obtain the permuted layer graph from the underlying graph H and the vertex permutation σ is to add the edge set $\{(\sigma(a), \sigma(b)) \mid (a, b) \in E(H)\}$ to H and then remove multiple edges. For example, Figure 1 shows a representation of K_6 as a permuted layer graph.

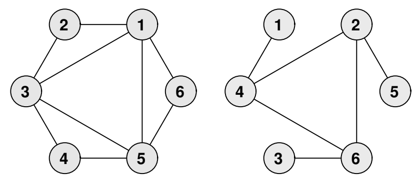


Figure 1: A permuted layer representation of K_6 using vertex label permutation (12)(34)(56). Edges (1,2), (3,4), and (5,6) are removed from the second layer.

Note that if $\tau \in \mathrm{Aut}(H)$ and $\sigma \in \mathrm{Perm}(V(H))$ then σ and $\sigma\tau$ produce isomorphic permuted layer graphs. Thus, the number of distinct permuted layer graphs with base H is at most $\frac{|V(H)|!}{|\mathrm{Aut}(H)|}$. However, if we restrict our analysis to full permuted layer graphs, the size of the set can be much smaller, as evidenced by the next proposition.

Proposition 1. *There is a unique full permuted layer graph whose base graph is the icosahedral graph.*

Proof. The labeling of the vertices of the icosahedral graph is given by the lefthand graph in Figure 2. The permutation of vertex labels given by $\sigma =$(2 10 3 7)(4 9 6 8)(5 11) yields a second layer that does not produce any multiple edges, as is shown in the righthand graph Figure 2. Thus there is at least one full permuted layer graph whose base graph is the icosahedral graph.

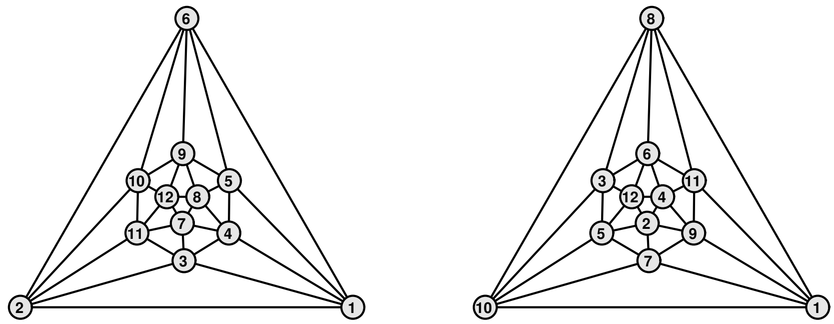


Figure 2: The unique full permuted layer graph whose base graph is the icosahedral graph.

On the other hand, let G be a full permuted layer graph whose base graph is the icosahedral graph H. Let σ be a vertex permutation that produces G from H. Since G is a full permuted layer graph the edge set $\{(\sigma(a), \sigma(b)) \mid (a, b) \in E(H)\}$ is disjoint from $\{(a, b) \in E(H)\}$. Thus every vertex in G has twice the degree it had in the icosahedral graph; that is, G has 12 vertices each of degree 10. Then G^c is a matching. Since all matchings on 12 vertices are isomorphic, all such G are isomorphic, and thus G is unique. $\square$

More generally, the class of permuted layer graphs contains all of the thickness-two graphs as subgraphs.

Proposition 2. *Every thickness-two graph is a subgraph of a permuted layer graph.*

Proof. Let H be a thickness-two graph with (edge-disjoint) planar layers H_1 and H_2. Suppose $|V(H)| = n$. Create a permuted layer graph, G, as follows:

Procedure **Permuted_Layer($H = H_1 \cup H_2$)**

1. Index the vertices of H (and hence of H_1 and H_2) by $v_1, v_2, \ldots, v_n$. Now re-index the vertices of H_2 by $v_{n+1}, \ldots, v_{2n-1}, v_n$ in order.
2. Identify H_1 and H_2 at v_n yielding a planar graph G_a. In G_a the vertices $v_1, \ldots, v_n$ induce H_1 and the vertices $v_n, \ldots, v_{2n-1}$ induce H_2.
3. Let $\sigma = (v_1\ v_{n+1})(v_2\ v_{n+1}) \ldots (v_{n-1}\ v_{2n-1})$. Note that σ is a product of

transpositions and that it fixes v_n.

4. Let G be the permuted layer graph obtained from the base G_a using the vertex permutation σ.

Note that the second layer of G is isomorphic to G_a but the vertex labels have been changed so that $v_1, \ldots, v_n$ induce H_2 while $v_n, \ldots, v_{2n-1}$ induce H_1. Therefore in G the vertices $v_1, \ldots, v_n$ induce both layers of H and thus H itself (as do the vertices $v_n, \ldots, v_{2n-1}$). Thus G is a permuted layer graph containing H as a subgraph. □

For example, the union of the two planar graphs in Figure 11 in Section 3.2, including edges (4,11) and (11, 12) (dashed) and excluding edge (4, 12) (bold), is a full permuted layer graph that contains Sulanke's graph as a subgraph.

3 New 9-critical Thickness-Two Graphs

To date there is only one published 9-critical thickness-two graph: Sulanke's graph S, due to the third author, is given by $S = K_6 \vee C_5$. An edge-disjoint decomposition of S as a permuted layer graph is shown in Figure 3. The K_6 is induced by vertices 1-6, and the C_5 is induced by vertices 7-11. In the next subsection, we introduce 40 new 9-critical thickness-two graphs.

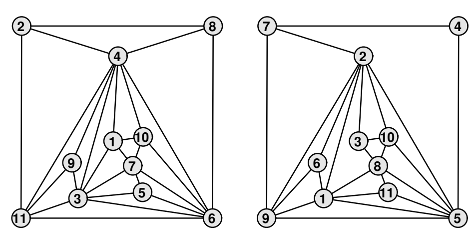


Figure 3: $K_6 \vee C_5$ thickness-two decomposition as a permuted layer graph.

3.1 Forty New Small 9-critical Thickness-Two Graphs

In this subsection we introduce 40 new 9-critical thickness-two graphs on 12 through 16 vertices. The graphs were selected from the hundreds found because they exhibit a wide range of values for the number of vertices and edges. Each graph G will be identified by its degree sequence and a reference number. The reference number is formated as $v.e.n$, where $v = |V(G)|$, $e = |E(G)|$, and n is a counter indicating that G is the n^{th} 9-critical graph on v vertices and e edges in our collection. For those subsets of graphs on the same number of vertices and edges (i.e., for which $n > 1$) we offer only such ones that have a unique degree sequence, which ensures that all graphs presented are distinct. Not all of the examples require a degree sequence for verification of uniqueness (those for which $n = 1$) but the labeling scheme is crafted with continued work and expansion in mind. Thus we give full information for each graph listed here.

The generation of 9-critical graphs that have few enough edges to be thickness-two was accomplished with a modified version of *geng*, which is a component of Brendan McKay's graph isomorphism software *nauty* [McK81, McK07a]. The thickness-two decompositions were found by randomly flipping diagonals of planar triangulations [Sul]. The pruning of those 9-chromatic graphs for which a thickness-two decomposition was found and that were not initially 9-critical was done with *nauty* as well. That each graph has thickness two was then verified with *Groups and Graphs* [KK05, McK07b] in order to display a thickness-two embedding for each example. The final graphics were generated by *Mathematica* [Wol07]. The 40 new 9-critical thickness-two graphs follow.

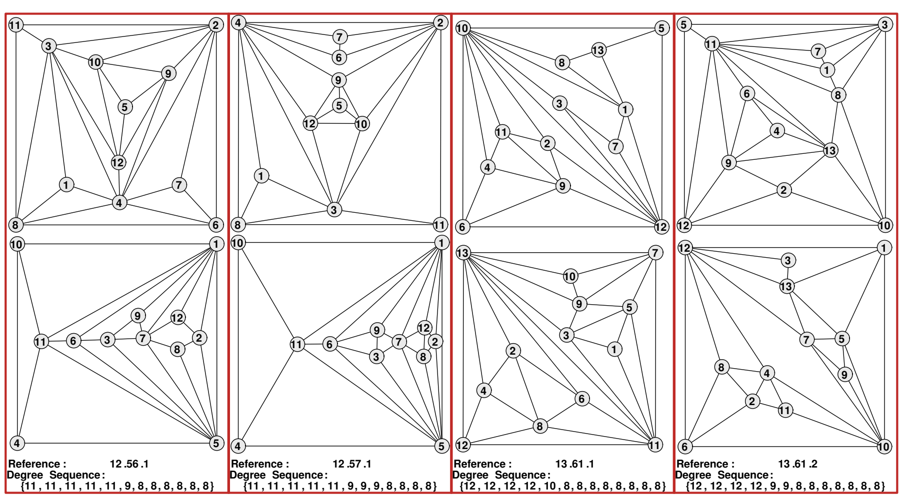


Figure 4: New 9-critical thickness-two graphs.

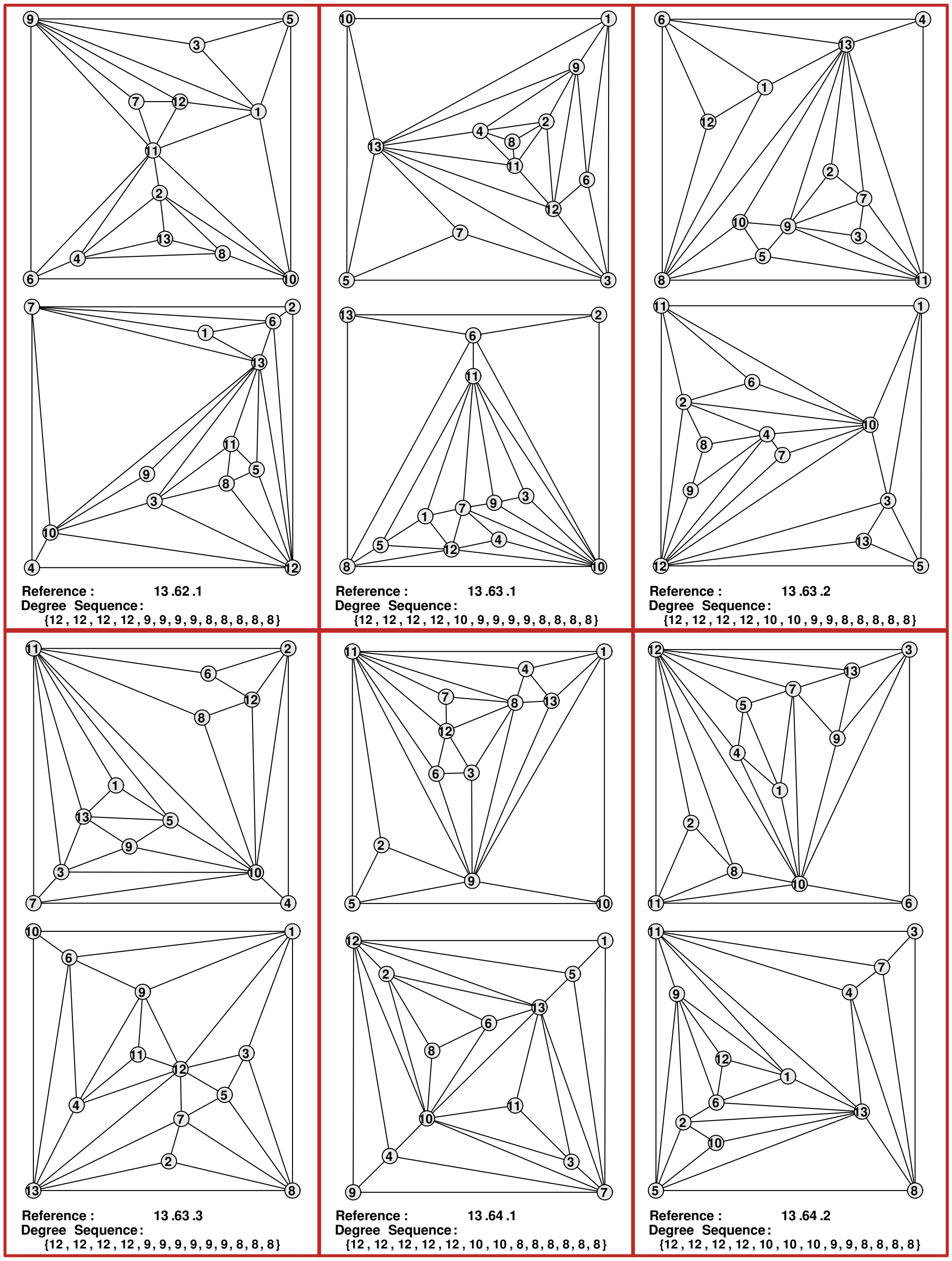


Figure 5: New 9-critical thickness-two graphs, continued.

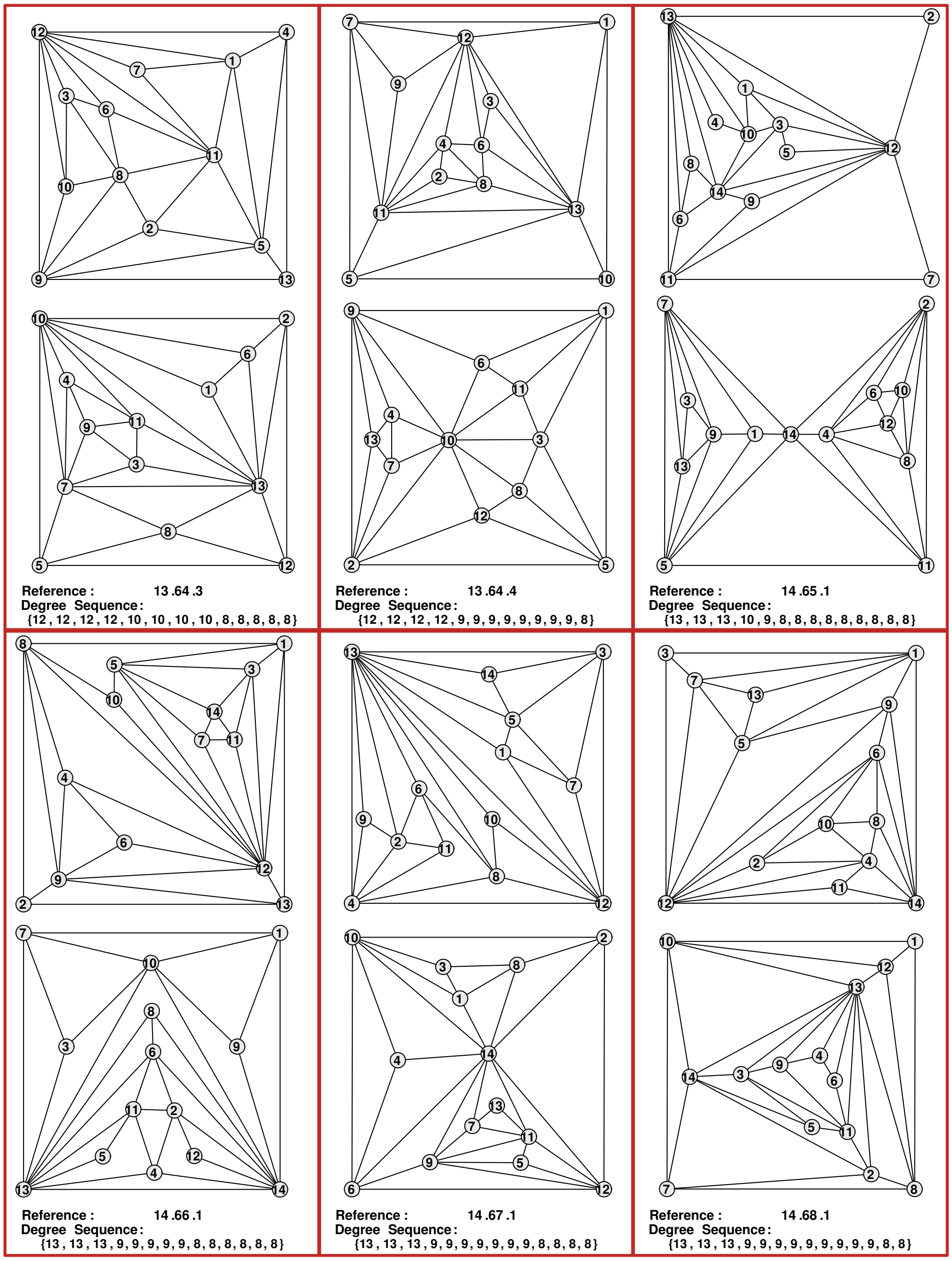


Figure 6: New 9-critical thickness-two graphs, continued.

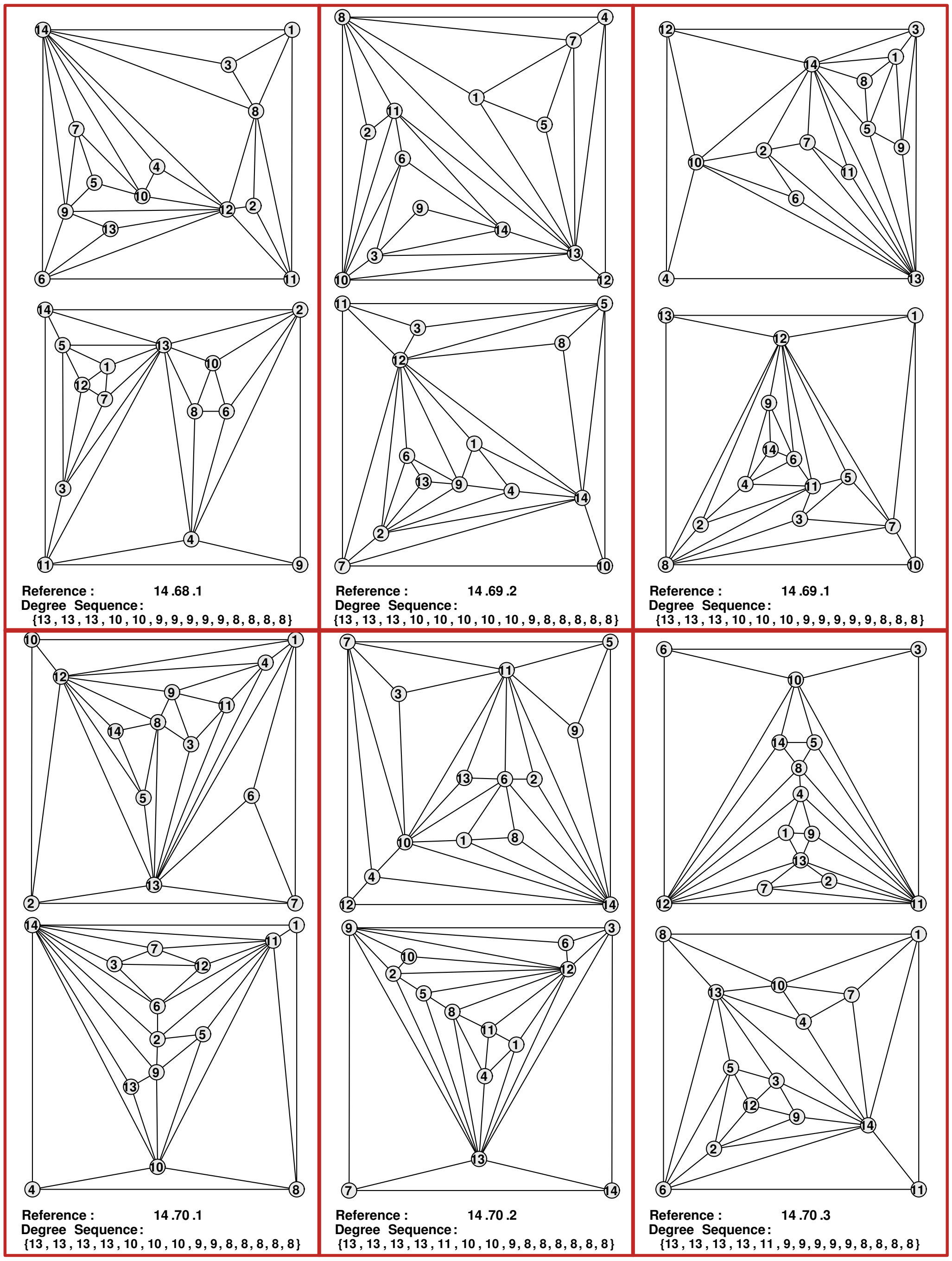


Figure 7: New 9-critical thickness-two graphs, continued.

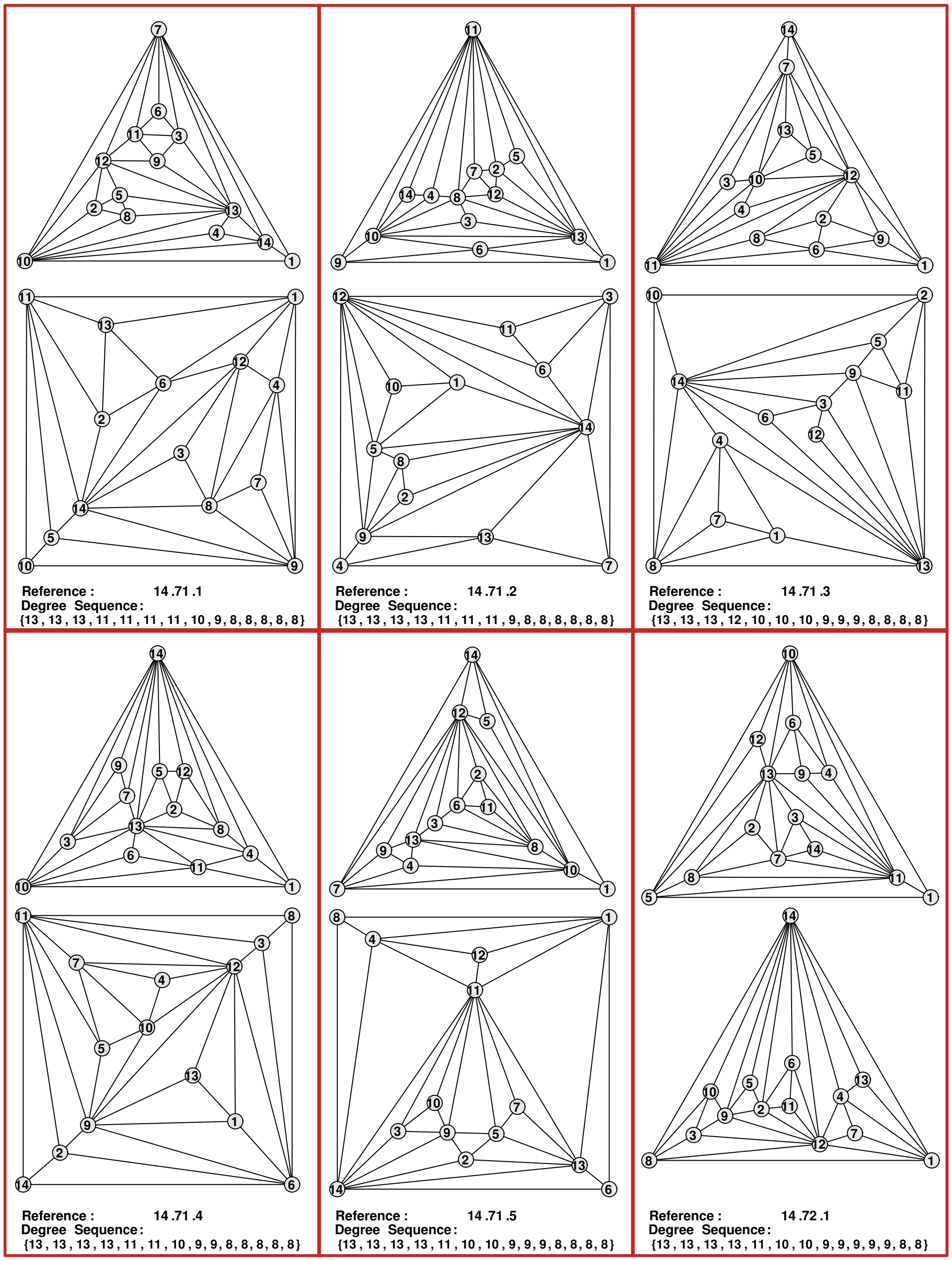


Figure 8: New 9-critical thickness-two graphs, continued.

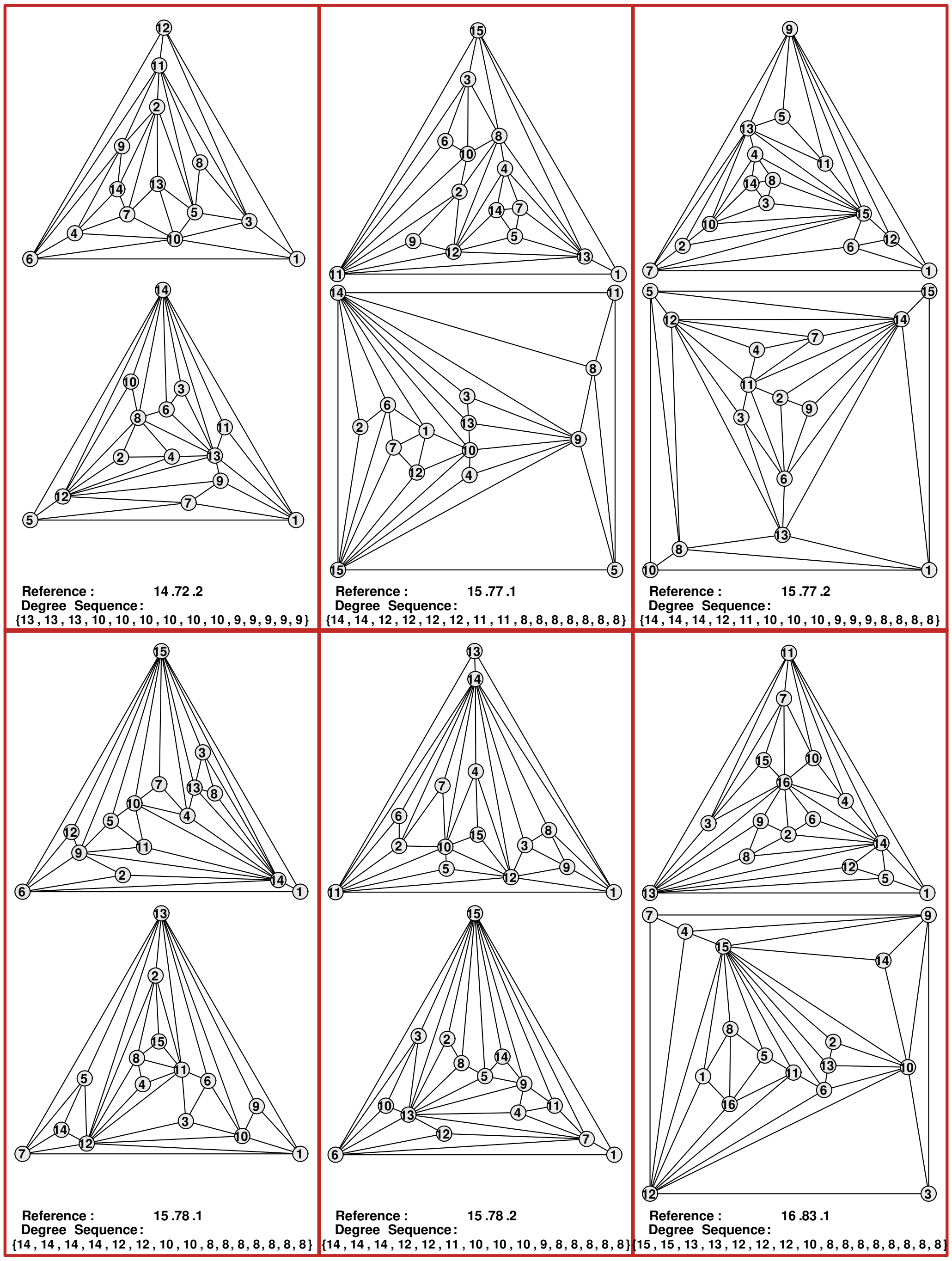
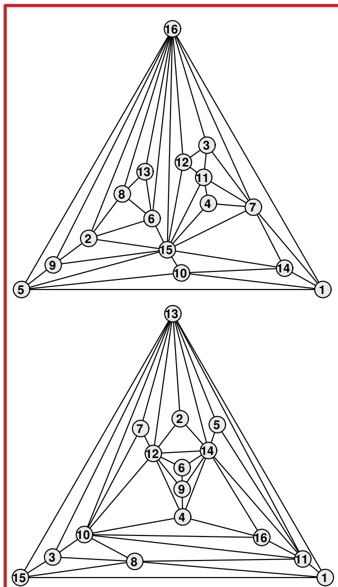
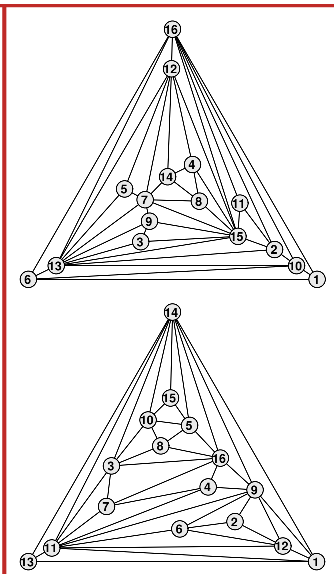


Figure 9: New 9-critical thickness-two graphs, continued.



Reference : 16 .84 .1
Degree Sequence :
{15 , 15 , 13 , 13 , 12 , 12 , 12 , 10 , 9 , 9 , 8 , 8 , 8 , 8 , 8 , 8}



Reference : 16 .84 .2
Degree Sequence :
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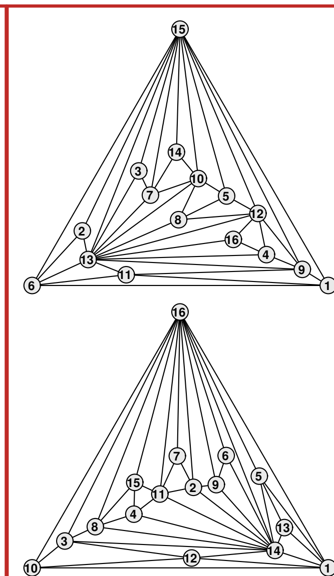
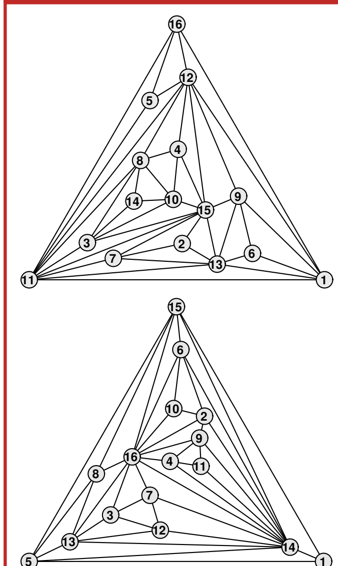


Reference : 16 .84 .3
Degree Sequence :
{15 , 15 , 15 , 15 , 11 , 10 , 10 , 10 , 10 , 9 , 8 , 8 , 8 , 8 , 8 , 8}



Reference : 16 .84 .4
Degree Sequence :
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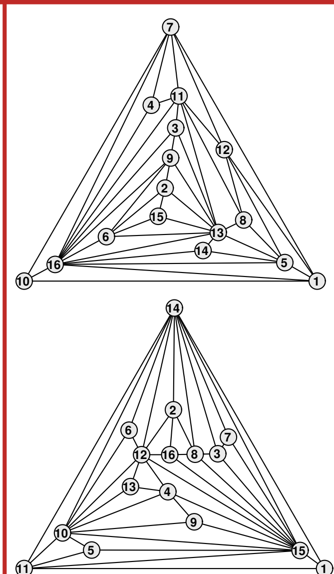

Reference : 16 .84 .5
Degree Sequence :
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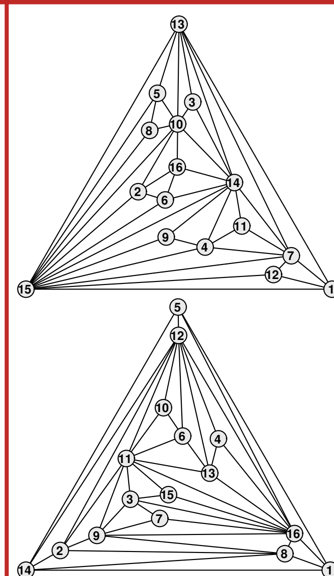

Reference : 16 .84 .6
Degree Sequence :
{15 , 15 , 15 , 12 , 12 , 12 , 11 , 10 , 9 , 9 , 8 , 8 , 8 , 8 , 8 , 8}

Figure 10: New 9-critical thickness-two graphs, continued.

3.2 Infinitely Many 9-critical Thickness-Two Graphs

In this section we provide a technique for constructing infinite families of k-critical thickness-two graphs. The construction requires two tools, one of which is the Procedure **Permuted_Layer** of Section 2, which is used in tandem with the well-known *Hajós construction* [Haj61, JR99, Urq97, Wes01, Wes83]. In addition, two (not necessarily distinct) k-critical thickness-two graphs are required to initialize the procedure. The Hajós construction is described next.

The Hajós Construction.

Step One: Let G_1 and G_2 be two k-critical graphs that are identified at a vertex v. If $(v, w_1) \in E(G_1)$ and $(v, w_2) \in E(G_2)$ then the graph $G = (G_1 - (v, w_1)) \cup (G_2 - (v, w_2)) \cup (w_1, w_2)$ is k-critical.
Step Two: Identification of any independent set of vertices in G yields a graph $\hat{G}$ such that $\chi(\hat{G}) \geq k$. The graph $\hat{G}$, however, is not necessarily k-critical.

Proposition 3. *Let H and K be graphs. Let G be the Step-One-Hajós of H and K. Then $\Theta(G) = \max\{\Theta(H), \Theta(K)\}$.*

Proof. Since G is the Step-One-Hajós of H and K, there exist vertices u, v, x so that x is the vertex at which H and K are identified in G, $u \in V(H), v \in V(K)$, $ux \in E(H), (v, x) \in E(K)$ in which case $G = H \cup K \cup \{(u, v)\} - \{(u, x), (v, x)\}$.

Suppose that $\Theta(G) = t$. Partition G into edge-disjoint planar layers $L_1, \ldots, L_t$ and assume without loss of generality that (u, v) is an edge of L_1. Since $L_1, \ldots, L_t$ are planar and x is a cut vertex of $G - \{(u, v)\}$, there are planar graphs $H_1, \ldots, H_t$ and $K_1, \ldots, K_t$ so that H_1 and K_1 identified at x yield $L_1 - \{(u, v)\}$, and for all $i > 1$, H_i and K_i identified at x yield L_i.

Notice that $H = (H_1 \cup \{(u, x)\}) \cup H_2 \cup \ldots \cup H_t$ and $K = (K_1 \cup \{vx\}) \cup K_2 \cup \ldots \cup K_t$. Further, if $H_1 \cup \{(u, x)\}$ is planar then there is a decomposition of H into t planar layers. Similarly, if $K_1 \cup \{(v, x)\}$ is planar, there is a decomposition of K into t planar layers. In particular, if both $H_1 \cup \{(u, x)\}$ and $K_1 \cup \{(v, x)\}$ are planar then $\max\{\Theta(H), \Theta(K)\} \leq \Theta(G)$.

Suppose to the contrary that $H_1 \cup \{(u,x)\}$ is not planar. Then by Kuratwoski's Theorem there exists a subgraph S of $H_1 \cup \{(u,x)\}$ that is a subdivision of K_5 or $K_{3,3}$. Since H_1 is planar, any such subdivision must include (u,x) as an edge.

Note that if K_1 is not connected we may add edges to make it connected while maintaining the planarity of L_1. Thus we may assume without loss of generality that K_1 is connected in which case there is a path P in K_1 from v to x. By replacing (u,x) by $(u,v)P$ in S we produce a subdivision of K_5 or $K_{3,3}$ in L_1. That is, $S' = S \cup P \cup \{(u,v)\} - \{(u,x)\}$ is a subdivision of K_5 or $K_{3,3}$ that is a subgraph of L_1, which is a contradiction since L_1 is planar. Thus $H_1 \cup \{(u,x)\}$ must be planar and so $\Theta(H) \leq t$. An identical argument yields $\Theta(K) \leq t$, which means $\max\{\Theta(H), \Theta(K)\} \leq \Theta(G)$.

It remains to show that $\max\{\Theta(H), \Theta(K)\} \geq \Theta(G)$. Suppose to the contrary that $\max\{\Theta(H), \Theta(K)\} = s < t$. Then each of H and K decompose into edge-disjoint planar layers $H_1, \ldots, H_s$ and $K_1, \ldots, K_s$ (note that some may be trivial since s is the maximum of two thicknesses). Define $L_1 = H_1 \cup K_1 \cup \{(u,v)\} - \{(u,x),(v,x)\}$. For $i = 2, \ldots, s$, define $L_i = H_i \cup K_i$. Clearly for $i > 1$, L_i is planar.

Since H_1 is planar and (u,x) is an edge of H_1 there is a plane drawing of H_1 with ux on the infinite face. Similarly there is a plane drawing of K_1 with (v,x) on the infinite face. Utilizing these particular drawings there is a plane drawing of H_1 and K_1 merged at vertex x with both (u,x) and (v,x) on the infinite face. Removing (u,x) and (v,x) and then adding (u,v) can be accomplished without any edge crossings. Hence L_1 is planar.

In that case $G = L_1 \cup \ldots \cup L_s$ is a decomposition of G into $s < t$ planar layers, which is impossible since $\Theta(G) = t$. Thus $\Theta(G) \geq \max(\Theta(H), \Theta(K)\}$. Altogether $\max\{\Theta(H), \Theta(K)\} = \Theta(G)$ as desired. □

The characterization given in Proposition 3 provides a means to a recursive algorithm to construct infinitely many 9-critical thickness-two graphs: all that is required is either one or two 9-critical thickness-two graph as the initial input.

Corollary 1. *There are infinitely many 9-critical thickness-two graphs.*

Penultimately for this section, we note that the pair of graphs in Figure 11 with the dashed edges (4,11) and (4,12) removed and the bold edge (4,12) included is an example of the Step-One-Hajós of two copies of the Sulanke graph.

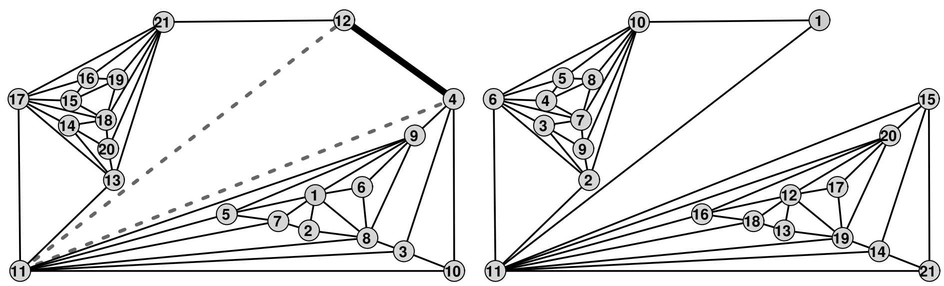

Figure 11: First step of the procedure given by the proof of Proposition 3 performed on two copies of the Sulanke graph produces a 9-critical graph on 21 vertices; include edge (4,12) (bold) and exclude edges (4,11) and (11,12) (dashed). The graph pictured is the first of an infinite family of such graphs.

Finally we note that the procedure outlined by the proof of Proposition 3 implies that if there is one example of a k-critical thickness-two graph, then there exist infinitely many k-critical thickness-two graphs, thus providing asymptotically good bounds for possible improvements to (1) in Section 1.

4 Catlin's Graphs

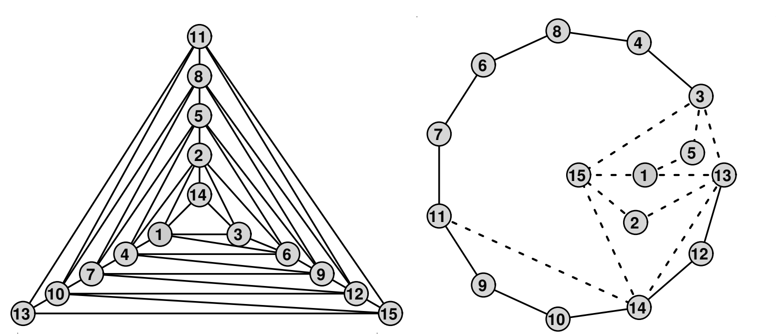

Figure 12: $C_n[K_3]$ has thickness two for all $n \geq 4$; $C_5[K_3]$ is pictured.

Typically, when one looks for high chromatic thickness-two graphs, a candidate graph is presented for which either the chromatic number or thickness, but not both, is known. Paul Catlin [Cat79] identified a family of graphs $C_n[K_r]$ in which each vertex of an n-cycle is expanded to a K_r and the join of "adjacent" copies of K_r is taken. Catlin's motivation was to show that there exist m-chromatic graphs that contain no subdivision of K_m. Toward this end he provides a formula for the chromatic number of $C_n[K_r]$. In this section, we show that every Catlin graph $C_n[K_3]$ with $n \geq 4$ has thickness two. This yields an infinite family of non-trivial graphs for which both the chromatic number and thickness are known. In particular, it is an easy consequence of [Cat79] that

$$\chi(C_n[K_3]) = \begin{cases} 8 & \text{if } n = 5 \\ 7 & \text{if } n \text{ is odd and } n \geq 7 \\ 6 & \text{if } n \text{ is even and } n \geq 4. \end{cases}$$

The thickness-two decomposition of $C_n[K_3]$ is described next.

Decomposition of $C_n[K_3]$ into two plane layers. In what follows, P_m is a simple path with m vertices. Let $G_n = C_n[K_3]$, and label the vertices of C_n by $v_1, v_2, \ldots, v_n$. We expand vertex v_i to a K_3 with vertices labeled by $\{3i-2, 3i-1, 3i\}$ for $i = 1, \ldots, n$. The edges of $C_n[K_3]$, in addition to those of each of the n copies of K_3, are given by the join of neighboring copies of K_3. The first layer of the thickness-two decomposition of G_n, written $G_{n,1}$, is given by the edge-disjoint union of three copies of P_n, n 3-cycles, and a P_{3n-2}. Specifically, the three P_ns are $\{1, 4, 7, \ldots, 3n-5, 3n-2\}$, $\{n-1, 2, 5, \ldots, 3n-7, 3n-4\}$, and $\{3, 6, \ldots, 3n-3, 3n\}$; the n 3-cycles are the subgraphs induced by vertices $\{3i+1, 3i-1 \pmod{3n}, 3i+3\}_{i=0}^{n-1}$; the P_{3n-2} is given by $\{3, 2, 1, 6, 5, 4, \ldots, 3i, 3i-1, 3i-2, \ldots, 3n-3, 3n-4, 3n-5, 3n\}$. Then $G_{n,1}$ is embedded in the plane by drawing the 3-cycles as concentric congruent triangles, and arranging the vertices so that the three P_ns are each on a line. The remaining edges are the ones belonging to the P_{3n-2}, each one of which is the diagonal of a face of size four; see the left graph in Fig. 2 for a plane drawing of $G_{5,1}$. Note that $|E(G_{n,1})| = 9n - 6$ and so $G_{n,1}$ is edge maximal since $|V(G_{n,1})| = 3n$.

Finally, the complement of $G_{n,1}$ in G_n, denoted $G_{n,2}$, is given by the edge-disjoint union of a P_{3n-4} and another planar graph G' with eight vertices and 11 edges. The latter can be embedded inside a plane drawing of the cycle induced by the P_{3n-4}. The order of the vertices in the path is given by $3, 4, 8, 6, 7, 11, 9, 10, \ldots, 3i-1, 3i-3, 3i-2, \ldots, 3n-1, 3n-3, 3n-2$. The remaining vertices of $G_{n,2}$ are $1, 2, 5$, and $3n$; the subgraph induced by these vertices together with vertices 3, $3n-2$, $3n-1$, and $3n-4$, is shown in dashed edges in the right graph in Figure 12. Note that $|E(G_n)| = 12n$, $|E(G_{n,1})| = 9n - 6$, and $|E(G_{n,2})| = 3n + 6$. Thus $|E(G_n)| = |E(G_{n,1})| + |E(G_{n,2})|$, as expected.

5 Open Questions

1. Figure 3 is a representation of Sulanke's graph as a permuted layer graph. We have shown by brute force that Sulanke's graph is not a full permuted layer graph; find a direct proof that this is so.

2. The permuted layer construction, as described in Section 2, requires as input a planar graph H on n vertices and a permutation of vertex labels $\sigma \in S_n$. The construction yields a thickness-at-most-two graph, which is the simple graph underlying $H \cup \sigma(H)$. Define $\beta(H)$ to be the number of distinct full permuted layer graphs generated by base planar graph H.

   (a) For which (planar) H is $\beta(H) = 1$? We know, for example, that if H is the icosahedral graph, then $\beta(H) = 1$; see Proposition 1.

   (b) What can be said of the following decision problem: given a positive integer k, is $\beta(H) = k$?

   (c) If $\sigma \in Aut(H)$ then $H \cup \sigma(H)$ is clearly planar. If H is maximal planar then $H \cup \sigma(H)$ is planar only if $\sigma \in Aut(H)$. For which planar graphs H and which $\sigma \notin Aut(H)$ is $H \cup \sigma(H)$ planar?

3. Are any of the 9-critical graphs in Section 3.1 (full) permuted layer graphs?

4. Step 2 of the Hajós construction identifies nonadjacent vertices. Doing so either preserves or increases the chromatic number. Find a criteria under which nonadjacent vertices can be identified while preserving or lowering the thickness.

5. For which $n \geq 4$, if any, is $C_n[K_3]$ a (full) permuted layer graph?

6. Find a 10, 11, or 12-chromatic thickness-two graph, or prove that none exists.

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