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Phase space orbits and the ping-pong ball impact oscillator

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We describe an inexpensive, readily assembled device, known as an impact oscillator, which can be constructed with commonly used introductory lab apparatus. It will allow the student to explore the use of phase space plots and to observe the behavior of a form of pendulum that is behaving nonlinearly. Using a motion detector, a teacher can demonstrate in simple, dramatic, and qualitative form the meaning and utility of concepts such as the phase diagram and bifurcation that are so important to the exciting field of nonlinear dynamics and chaos. We include a technique for improving the plots produced by the motion detector software for the case of periodic motion.

Notions of chaotic behavior burst into the popular consciousness roughly two decades ago, aided by materials targeted for the general audience. Part of the scientific community had been hard at work on it for several decades before, beginning with a surge of activity initiated by E. N. Lorenz. A smaller number of mathematicians and physicists had been studying the problem since Henri Poincaré’s investigations into the celestial three body problem at the end of the nineteenth century. Comparatively recently some ingenious and relatively simple but quite intriguing table top experiments have been devised to study chaos in the classroom and teaching laboratory.

Assembling the impact oscillator

The impact oscillator uses commonly available lab equipment: a computer controlled motion detector, a loudspeaker, a signal generator, a power amplifier, and a pendulum. It can be used in a number of ways, ranging from a quick introductory physics demonstration intended to introduce a new and useful way to plot motion, to an open ended project at an advanced level. The apparatus, the heart of which is shown in Fig. 1, consists of a ping-pong ball pendulum driven by a loudspeaker modified to serve as a mechanical driver.

To make the pendulum, the center of a length of thread about 100 cm long is glued to a ping-pong ball along a 7 to 8 mm stretch on the seam of the ball. The ends of the thread are tied to points about 10 cm apart on a horizontal support rod with the ball at the point of the vee. The effective length of our pendulum is approximately 45 cm. A 2 cm long, 1 cm diameter nylon rod is hot

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glued to the cone of a speaker and a driver plate, consisting of a 3 cm square of .2 cm thick glass (these dimensions are approximate and not at all critical), is glued to the end of the rod. In operation, the signal generator/amplifier output is adjusted so that the motion of the driver plate will typically be on the order of 1 to 2 mm over a span of frequencies from 4 Hz to 8 Hz. The entire apparatus is set up on a base of sturdy plywood so it can be tilted, eventually, using a suitable block of wood beneath one end of the base. The stands used to support the pendulum, speaker-driver, and motion detector should all be clamped to the base so the apparatus can be tilted as a unit without changing the relative position of the components.

The phase space plot
A central notion in the study of complex mechanical motions is that there are special advantages connected with how we display the data describing the motion. The usual motion detector software easily permits us to plot velocity versus position. Known as a phase space plot, it has the property that any point on it, i.e., a particular value of velocity and position, can be seen as the initial conditions for motion that takes place after that point. Then only one curve should pass through that point since a particular set of initial conditions should result in only one particular motion. If, on the contrary, two curves passed through a point, the implication is that there could be two different motions for the same initial conditions! Thus, the phase space plots for a ball tossed upward under the influence of a constant gravitational force typically look like those in Fig. 2. The difference between the two plots is the initial upward velocity. Our point here is that the experimenter cannot launch the ball in such a way that the curves will cross (for the exercise, students can be asked to devise a counter-example). There is a caveat—if gravity’s acceleration, $g$, depended on time so that when the second launch occurred, $g$ was weaker or stronger, then the curves could cross. This would be like doing your ball tossing experiment on a variably accelerating elevator! However, there is a large class of interesting problems where, for example, the forces are constant or depend on position—as in the case of the spring or the pendulum—but do not depend explicitly on time.

![Velocity versus Height](image)

**Fig. 2.** Phase space for constant gravitational acceleration.

This kind of statement about how curves in phase space shouldn’t cross is an example of the approach taken in the analysis of systems far more complex—read that as nonlinear—than the one-dimensional uniform gravitational problem. The point is, and this was a major result of Poincaré’s work, that even when it isn’t possible to express the solution in simple terms, it may still be possible to make useful statements about the geometry of the system’s behavior in phase space.

The phase space plot for a pendulum
Turning to our pendulum and motion detector apparatus, we note that, with the
wooden base of the apparatus horizontal and the speaker-driver out of the way, the ping-pong pendulum can oscillate without hitting it and we get the phase space plot shown in Fig. 3.

If this part of the demonstration is done with a solid wooden or rubber spherical bob, so that the damping is less obvious than in the case of the ping-pong ball, the tracks look like ellipses. Some natural conceptual teaching points: Where is the pendulum in its arc for an instructor-chosen point on the phase space plot? How can you vary the initial conditions to obtain different ellipses? When damping is present will the paths from different initial conditions cross? Finally, a discussion of the conservation of energy can be used to show that, in the absence of dissipative forces, the phase space path for small swing amplitude is an ellipse.

The reflected pendulum

Next, we place a rigid vertical barrier, a flat piece of wood or aluminum plate, so the pendulum bob strikes it at the bottom of its swing and is reflected. In a fully elastic collision, the effect would be to reverse the momentum of the bob without changing its magnitude. In fact the bob will lose energy two ways: upon collision with the barrier and because of air resistance. Data from the motion detector for the reflected pendulum is shown in Fig. 4.

The motion detector samples the position of the bob every 1/30 of a second, relatively slow compared to the time for a collision. Since the data collection software finds an average velocity based on three or more consecutive position values, the number depending on a choice the user makes, the effect is to round off the phase space plot in the region of the collision. An idealized phase space plot would have a vertical line connecting the negative and positive values of the velocity at the position of the wall. In the course of a demonstration, it is easy to see the difference between the trace representing the smooth, gravity controlled motion and that part affected by the collision.

Running the impact oscillator—a driven reflected pendulum

To set up the impact oscillator we locate the speaker-driver in place of the vertical barrier so the pendulum bob just touches the motionless driver plate when the pendulum
is vertical and at rest. Next, we tilt the base to an angle of approximately 10 degrees so that the swing between successive collisions with the speaker-driver will take longer for larger amplitude of the swing.

By tilting the base roughly 10 degrees we retain a motion that is slow enough for the motion detector to follow, we allow the driver to produce a swing that is large enough to be readily measured, and we obtain a mild amplitude dependence of the time of flight between collisions. The latter turns out to be important for the production of mixtures of large and small swings seen below.

With speaker-driver frequency of 5.8 Hz, frequency generator amplitude of approximately 1.30 V peak-to-peak, and a hand-launched initial displacement of the pendulum from the driver of about 5 to 10 cm, we see the phase space orbit shown in Fig. 5. The graph title “Single bounce” refers to the presence of just one amplitude for the swing of the pendulum. Several starts may be necessary to get the pattern shown. There is a brief wait for the motion to stabilize.

We have seen two modes of oscillation: one where the bob is struck once per driver cycle and one where the bob is struck every other driver cycle. The two modes correspond, in general, to smaller and larger amplitudes, respectively. Recall that a larger amplitude oscillation of the tilted reflected pendulum has a longer period. Under the conditions leading to Fig. 5, a single, large amplitude orbit (swing out and in) of the pendulum takes twice the period of the driver. The velocity calculations distort the orbit near collision as in Fig. 4.

**Fig. 5. Typical phase space path for the impact oscillator.**

**Bifurcation and period doubling**

By starting with the system running in the single bounce mode, as in Fig. 5, and increasing the voltage output of the frequency generator, we arrive at the pattern in Fig. 6 at frequency generator amplitude of 1.44 V peak-to-peak. As the drive voltage is increased beyond 1.30 V peak-to-peak the phase space path thickens, splits, and the split increases with increased drive voltage. The qualitative shift in behavior between Fig. 5 and Fig. 6 is called a bifurcation. The pendulum period is now four times the driver period: the time for one complete circuit of the phase space orbit is doubled compared to the time for completing the orbit in Fig. 5. The large bounce in Fig. 6 is larger than the bounce in Fig. 5 and takes somewhat longer. However, the small bounce takes a little less time than that in Fig. 5 so the time to complete the orbit in Fig. 6 is exactly twice what it took for the orbit in Fig. 5. Direct visual observation shows that large and small bounces are alternating. We can describe the impact oscillator as producing two different loops in
phase space by alternately hitting the bob sharply (causing the big loop) and softly (causing the small loop), depending on when during the cycle of the driver’s motion the collision occurs.

Increasing the drive voltage to 1.53 V peak-to-peak produces Fig. 7. We see that the inner and outer loops are doubled. There are orbits with longer period and more complex structure, but they are unstable and it is necessary to avoid air currents or vibrations of the supporting table if they are to be seen. We have also looked at the mode for smaller orbits and have found interesting behavior there as well. For example, we have seen sequences of one, three, and six different bounce amplitudes per orbit.

**Improving the plots**

Images like those in Figs. 5, 6, and 7 are easy to get from the software used to control the motion detector and especially convenient in a lecture-demonstration setting. We can improve these plots by taking advantage of the periodicity of the motion. If we examine a plot of a few cycles of position versus time, Fig. 8, for the settings that led to Fig. 5, we see that the sampling rate is low compared to the period of oscillation, and the data scarcity problem is pronounced near the collision where the pendulum moves most rapidly.

If we plot points according to where they fall within the period of the motion, rather than according to the time at which they were measured, we fold all the data for, say, a 10 second span into a single period, then the information becomes dense as can be seen in Fig. 9.
Ten seconds of data processed in this way has yielded 300 points in the period of about .35 second. We can now follow the movement from just after collision to just before it. To plot all data in one period we export the time and position data from the data collection software to a spreadsheet, Excel in our case, to take advantage of the latter’s flexibility and function set. Using the modulo function, MOD, we generate a column of reduced times by evaluating MOD(time – shift, period) for every time value. “Time” refers to the original time for each position measurement. “Shift” is a constant for all data points that allows us to set the time origin to coincide with the zero of position. “Period” is a constant that is the inverse of the frequency of the pendulum motion. Note that “period” is twice the period of the driver if, as is the case for Fig. 5, the driver hits the pendulum on every other outward stroke. For convenience in adjusting the plots, “shift” and “period” are stored in separate cells and referred to using absolute cell reference in the MOD function.

Using Excel Solver, we can fit the folded plot in Fig. 9 with a quadratic function of time and find the curve to be within a reasonable limit of error (less than \( \_\text{mm}^{16} \)) of the data across the entire period. The quadratic and its derivative are combined to create a phase space plot, as seen in Fig. 10 with the data from Fig. 5 superimposed for comparison. The calculated phase space path is closed by a vertical line connecting the uppermost and lowermost points at the 0 cm position.

**Conclusion**

We have described how to fashion and use a simple impact oscillator whose motion is observed using a widely available ultrasonic motion detector and its associated control software. The inexpensive apparatus can be incorporated into a lecture on chaos for an introductory physics audience and into the introductory mechanics lab. Students can learn how phase space diagrams are used and they can observe phenomena such as bifurcation and period doubling. There are a number of features of the apparatus that can be studied by those who wish to delve more deeply. The parameter space spanned by drive voltage and drive frequency can be explored to see what kind of behavior is accessible in different regions. Values for model parameters such as the coefficient of
restitution, air resistance, and the effective
driver-to-bob mass ratio can be established.
It is useful and interesting to make a detailed
study of stability and the phase of the driver
at the moment of impact as operating
conditions are varied. A computer model
for the impact oscillator can be developed
and compared with observed behavior.

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8 The Ultrasonic Motion Detector, Logger Pro software, and the LabPro Interface are all products of Vernier Software and Technology, 13979 S.W. Millikan Way, Beaverton, OR  97005.
9 4” replacement speaker, wide range, 8 ohm, 3 watts, Radio Shack Corporation, http://www.radioshackcorporation.com/
14 See James Gleick, p. 69, op. cit., for a discussion of bifurcations as they appear in population dynamics and R. C. Hilborn, p. 11, op. cit., for an example drawn from electronics.
15 If the pendulum frequency was 3 Hz and the sampling frequency is 30 Hz we would see the same 10 points in every swing of the pendulum. This is easily avoided by adjusting the data sampling rate or the frequency.
16 See MacIsaac and Hamalainen, p. 42, op. cit.