Propagation of Error

When doing experiments, we often measure two or more distinct physical quantities and then combine the measurements mathematically to obtain some final result. For example, to determine the average speed of an object we could measure the amount of time it takes to travel a certain distance and then use

$$\text{average speed} = \frac{\text{distance traveled}}{\text{elapsed time}}$$

The uncertainty in the calculated average speed depends upon the uncertainty in the distance as well as the uncertainty in the time. Propagation of error refers to the methods used to determine how the uncertainty in a calculated result is related to the uncertainties in the individual measurements.

The methods used for determining how uncertainties are combined can be quite sophisticated. In this course we will use approximate methods that are easier to use, but still give reasonable agreement with more advanced methods.

THE MIN-MAX METHOD

The basic method we will use to propagate errors is called the min-max method. To use this method we define a minimum and maximum value for each of the measurements used to calculate the final result. The minimum and maximum values are simply (best value - uncertainty) and (best value + uncertainty). Then we use these values to calculate a minimum and maximum value for the calculated result. From the minimum and maximum values for the calculated result, we deduce the uncertainty in the result. Here are some examples.

1) Addition of measurements

When two resistors are connected in series, the total resistance is the sum of the individual resistances. Suppose that two resistors in series have resistances of 78 ± 1 ohms and 135 ± 2 ohms. The best estimate of the total resistance is 78 + 135 = 213 ohms. Based on the uncertainty, it is quite likely that the resistance of the first resistor is between 77 and 79 ohms and the resistance of the second resistor is between 133 and 137 ohms. These are our minimum and maximum values. Using the maximum values for each resistor we get a maximum total resistance of 79 + 137 = 216 ohms. Similarly, the minimum total resistance is 77 + 133 = 210 ohms. The difference between the maximum resistance and the minimum resistance is 6 ohms so the total resistance with uncertainty is 213 ± 3 ohms. In this case the total uncertainty is just the sum of the two uncertainties.

Notice that we have used the terms minimum and maximum a little carelessly. For example, 78 ± 1 ohms does not mean that the resistance is guaranteed to be within 77 and 79 ohms. Instead it means that there is about a 68% probability that the resistance falls within that range. The same is true for the other resistor and for the total resistance. For the remainder of this handout we will use the terms maximum and minimum, but we must always keep in mind that these are not rigid limits on the values.

2) Subtraction of measurements

The mass of an empty thermos bottle is found to be 78.3 g ± 0.2 g. When filled with liquid nitrogen its mass is 167.7± 0.3 g. The best estimate for the mass of liquid nitrogen in the flask is 167.7 - 78.3 = 89.4 g. The minimum and maximum values for the mass of the thermos bottle are 78.1 g and 78.5 g. The minimum and maximum values for the mass of the thermos bottle plus nitrogen are 167.4 g and 168.0 g. To find the maximum mass for the liquid nitrogen we subtract the *minimum* mass of the thermos bottle, 78.1 g, from the maximum mass of the thermos + liquid nitrogen, 168.0 g. The result is that the maximum value for the mass of liquid nitrogen is 168.0 g - 78.1 g = 89.9 g. To find the minimum mass of the liquid nitrogen we subtract the *maximum* mass of the thermos, 78.5 g, from the minimum mass of the thermos + nitrogen, 167.4g. The result is 167.4 g - 78.5 g = 88.9 g. Thus the mass of the liquid nitrogen is 89.4 ± 0.5 g. Note that the masses are subtracted, but the uncertainties *add*.

3) Multiplication by a constant

The time for a pendulum to complete 5 swings is found to be t = 12.6 ± 0.2 s. The time for 1 swing, T, is given by T = (1/5)t. The best estimate for T is (1/5) x 12.6 = 2.52 s. The maximum time for one swing is (1/5) x 12.8 = 2.56 s and the minimum time is (1/5) x 12.4 s = 2.48 s. Thus the time for one swing is T = 2.52 ± 0.04 s. Note that the uncertainty is 1/5 of the uncertainty in the original measurement.

4) Multiplication of measurements

A rectangular plot of land is found to be 163 ± 1 ft by 386 ±2 ft. The best estimate for the area of the plot is 163 ft x 386 ft = 62918 ft^2. The maximum area is 164 ft x 388 ft = 63632 ft^2. The minimum area is162 ft x 384 ft = 62208 ft^2. The maximum area is larger than the best estimate of the area by 714 ft^2. The minimum area is smaller than the best estimate by 710 ft^2. Taking the average we have that the area is 62918 ± 712 ft^2. We recall that uncertainties are normally quoted to only one or two significant figures, and the precision of the result should match that of the uncertainty. Thus the uncertainty is 700 ft^2, and the result is 62900 ± 700 ft^2. It is worth noting that if we were using rules for significant figures to decide how to round off the answer, we would give the answer to 3 significant figures, namely 62900 ft^2. With our analysis we not only find out how to round off the final result, but we also get information about the actual size of the uncertainty.

5) Division of measurements

A sprinter runs 400 ± 2 m in 65.31 ± 0.05 s. Her average speed is 400m/65.31s = 6.12464 m/s. To find the maximum value for her speed we take the maximum distance and the *minimum* time (similar to what we did with subtraction above) and get a speed of 402m/65.26s = 6.15998 m/s. The minimum value for her speed is 398m/65.36s = 6.08935 m/s. The maximum and minimum are both different from the best value by about 0.035 m/s. Rounding to 1 sig fig we get that the uncertainty is 0.03s. Rounding the best estimate properly we have an average speed of 6.12± 0.03 m/s.

6) Raising to a power and multiplying by a constant

The radius of a circle is found to be 7.5 ± 0.1 cm. The best estimate for the area of the circle ($A = \pi R^2$) is A = 3.1416 x 7.5^2 = 176.715 cm^2. The maximum area is 3.1416 x 7.6^2 = 181.459 cm^2 and the minimum area is 3.1416 x 7.42 = 172.034 cm^2. The difference between the maximum value and the best estimate is 4.744 cm^2 and the difference between the minimum and the best estimate is 4.681 cm^2. Thus the uncertainty is about 4.7 cm^2, or to 1 significant figure, 5 cm^2. The final result for the area is A = 177 ± 5 cm^2.

RULES FOR PROPAGATING UNCERTAINTIES

The above examples illustrate how the min-max method may be applied in a number of specific cases. We can also apply the method more generally and develop some simple rules for propagating errors. We will not work through the derivations now, but will instead just present the results. The derivations may be found in any error analysis textbook and you will be learning them next semester.

Rule #1

When two measurements are added or subtracted, the absolute uncertainty in the result is the sum of the absolute uncertainties of the individual measurements.

Rule #2

When a measurement is multiplied by a constant, the absolute uncertainty in the result is equal to the absolute uncertainty in the measurement times the constant, and the relative uncertainty in the result is the same as the relative uncertainty in the measurement.

Rule #3

When two measurements are multiplied or divided, the relative uncertainty in the result is the sum of the relative uncertainties in the individual measurements. This rule could be stated equivalently in terms of percentage uncertainties, since relative and percentage uncertainties are simply related by a factor of 100.

Rule #4

When a measurement is raised to a power, including fractional powers such as in the case of a square root, the relative uncertainty in the result is the relative uncertainty in the measurement times the power.

Comments:

Clearly, Rule #1 agrees with examples 1 and 2 given for the min-max method.

Rule #2 also obviously agrees with the result from example 3, but it contains the extra information that the relative error remains the same when a quantity is multiplied by a constant. Checking this we see that the relative error in the original time measurement is 0.2 s/12.6 s = 0.0159, and the relative error in the time for one swing is 0.04 s/2.52 s = 0.0159.

Now we will apply Rule #3 to example 5 - division of two measurements - to check that it gives the same result. The relative uncertainty in the distance run is 2 m/400 m = 0.005. The relative uncertainty in the time taken is 0.05 s/65.31 s = .00077. Adding the relative uncertainties we get 0.005 + 0.00077 = 0.00577. According to Rule #3, this is the relative uncertainty in the speed. However, we want to know the *absolute* uncertainty in the speed. Since

$$\text{relative uncertainty} = \frac{\text{absolute uncertainty}}{\text{best estimate}},$$

we can rearrange to get

$$\text{absolute uncertainty} = \text{relative uncertainty} \times \text{best estimate}.$$

Thus we find that the absolute uncertainty in the speed = 0.00577 x 6.12464 m/s = 0.035 m/s. This is the same answer that we got in example 5.

In addition to giving the same answer, application of Rule #3 has another benefit. By calculating the relative errors in the measurements we obtain information about which measurement contributes most to the uncertainty in the result. In this example we see that most of the uncertainty in the speed is due to the uncertainty in the distance measurement. This type of information is useful when trying to decide how to improve an experiment or when trying to sort out why an actual result does not agree with an expected result. Often, to simplify our calculations, we will look for the most significant source of error in an experiment and disregard smaller sources of error.

Now we will apply Rules #2 and #4 to example 6. According to Rule #2, multiplying by a constant does not change the relative error, and according to Rule #4, the relative error in R^2 is 2 times the relative error in R. Thus we have that the relative error in the area of the circle is twice the relative error in the radius. The relative error in the radius is 0.1 cm/7.5 cm = 0.0133. Multiplying by 2 we get that the relative error in the area of the circle is 0.0266. The absolute error in the area is 0.0266 x 176.715 cm^2 = 4.7 cm^2, which agrees with the earlier result.

One final word about the methods we will use to find uncertainties. As stated earlier, these methods are approximate. More sophisticated methods for finding uncertainties will typically give results equal to, or smaller than those obtained by our methods. In other words, the methods we use tend to overestimate the uncertainty. The reason is that we have assumed a worst-case scenario in which errors from two measurements always add together. In reality, there is a chance that the errors from two independent measurements will partially cancel. More sophisticated methods of error analysis take this possibility into account. However, the difference is often not very important. In many cases we will make only rough estimates of the uncertainty in our measurements and so even if we were to apply fancier methods of error propagation, we would not have a lot of confidence in our final uncertainty. In this course, what we want to achieve is a rough idea of the uncertainty, say to within a factor of 2. Thus the approximate methods described in this handout are quite sufficient.