

Laissez les bonnes ondes rouler! (Let the good waves roll!)

Here-in lies a too-brief review of the highlights of the material we discussed this spring...

1. PRE-WAVES:

1.1. Taylor Expansion and Universality of SHM. Any analytic function - most importantly for us *potential energy* $U(x)$ - may be expanded around any point $x_o = x + \Delta x$ as

$$U(x + \Delta x) = U(x) + \left. \frac{dU}{dx} \right|_x \Delta x + \frac{1}{2} \left. \frac{d^2U}{dx^2} \right|_x \Delta x^2 + \dots \quad (1)$$

The first physically important, non-vanishing term for a system moving in the neighborhood of a stable equilibrium ($dU/dx = 0$) point is the second derivative. This is the effective spring constant for oscillations around this point

$$k_{eff} = \left. \frac{d^2U}{dx^2} \right|_{x=x_o}.$$

The small angle approximations are also derived with this e.g. $\sin \theta \approx \theta$.

1.2. Equations of Motion. Newton's

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$$

gives equations of motion. These are second order differential equations (e.g. see equation 2). For a unique result we need two *initial conditions* per independent variable (e.g. initial position $x(0)$ and initial velocity $v(0)$).

1.3. Potential Energy. The potential energy $U(x)$ of a system determines the local and nature of equilibria. The force is related to the slope of the potential via

$$\vec{F}_x = -\frac{\partial U}{\partial x} \text{ or in general } \vec{F} = -\nabla U.$$

So we can see that the first derivative dU/dx vanishes at equilibria when $F = 0$. These equilibria are stable, unstable, or neutral if d^2U/dx^2 is greater than zero, less than zero, or vanishing, respectively.

1.4. Simple Harmonic Motion (SHM). Simple harmonic motion is a universal behavior of systems around stable equilibria (derived via the potential energy and the Taylor expansion above). We found that the SHM the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad (2)$$

Motion is characterized by ω or the period $T = 2\pi/\omega$. For a mass on a spring $\omega = \sqrt{k/m}$ while a pendulum has $\omega = \sqrt{g/l}$. The solutions of the equation of the motion can be written as

$$x(t) = x_m \sin(\omega t + \varphi) \quad (3)$$

(cosine works just as well) where the *amplitude* x_o and *phase* φ are fixed by *initial conditions*, e.g. $x(0)$ - where is the mass at $t = 0$? - and $dx/dt(0)$ - How fast is it moving at $t = 0$? These solutions satisfy

superposition so we can add them. (This property is a result of the linearity of the equation of motion.) SMO's are a great way to store energy. The total energy of an oscillator is

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_o^2 = \frac{1}{2}m\omega^2x_o^2 \quad (4)$$

SHM occurs in many systems including any linear media¹ like Jello! By the universality of SHM argument we found (as above)

$$k_{eff} := \left. \frac{d^2U}{dx^2} \right|_{x_o}$$

when x_o is a stable equilibrium point. If you know the potential energy this is the most direct way to obtain the “frequency of small oscillations”.

1.5. Damped oscillation. A “damped” oscillator has a linear velocity-dependent damping force, $\mathbf{F}_d = -b\mathbf{v}$ has the equation of motion

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_o^2 x = 0 \quad (5)$$

We used $\beta = b/(2m)$ so the equation of motion becomes

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_o^2 x = 0 \quad (6)$$

For lightly damped cases ($\beta/\omega_o \ll 1$), this equation of motion has solutions

$$x(t) = x_m e^{-\beta t} \sin(\omega t + \varphi).$$

While x_m and phase φ are set by initial conditions, $\omega = \sqrt{\omega_o^2 - \beta^2}$. The now-time dependent amplitude, $x_m e^{-\beta t}$, is the envelope into which the oscillations, $\sin(\omega t + \varphi)$, are fit.

1.6. Damped, driven oscillation. An oscillator with both a velocity-dependent damping force, $-b\mathbf{v}$, and a driving force with angular frequency ω , $f_o \cos(\omega t)$, has the equation of motion

$$\begin{aligned} \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_o^2 x &= \frac{F_o}{m} \cos(\omega t) \text{ or} \\ \frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_o^2 x &= \frac{F_o}{m} \cos(\omega t) \text{ with } \beta = \frac{b}{2m}. \end{aligned} \quad (7)$$

We used $\beta = b/(2m)$ to characterize the solutions. For *lightly damped motion* ($\beta/\omega_o \ll 1$), this equation of motion has late time solutions

$$x(t) = A(\omega) \cos[\omega t + \varphi(\omega)] \quad (8)$$

where the *amplitude* $A(\omega)$ depends on the driving angular frequency ω

$$A(\omega) = \frac{F_o}{m} \frac{1}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad (9)$$

and the *phase* $\varphi(\omega)$ is

$$\varphi(\omega) = \arctan \left[\frac{2\beta\omega}{(\omega^2 - \omega_o^2)} \right] \quad (10)$$

¹Satisfying

$$\frac{\Delta L}{L} = \frac{1}{Y} \frac{F}{A}$$

where Y is Young's modulus. We saw that jello jiggled with SHM (in this case it was shear motion and Y was replaced by the shear modulus). Volume deformations around equilibrium satisfy

$$\frac{\Delta V}{V} = -\frac{1}{B} \Delta P$$

with B being the bulk modulus. We used this to derive the wave equation for sound.

Damped, driven oscillators are characterized by the *quality factor* Q , the fractional energy dissipation per radian or

$$Q \approx \frac{\omega_o}{2\beta} = \frac{m\omega_o}{b} \approx \frac{f_o}{\Delta f}.$$

The characteristic feature of damped, driven oscillators is *resonance* which is the (often catastrophic) growth in amplitude x_o as the driving frequency approaches $\sqrt{\omega_o^2 - b^2/2m}$.

Nature has a way to save itself from resonant destruction - *waves*. Waves carry potentially destructive energy away from the oscillator. This is often accomplished by interactions of the system with its environment. We derived the resulting *wave equation* (see section 3) for waves on a string and for sound. Light also satisfies the wave equation; light is a wave.

1.7. Fourier Series. Fourier's theorem says that essentially all functions can be written as a sum of sines and cosines. It is prescriptive so we know how to do this. For a function $f(x)$ compute 'overlaps'

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

and

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

and then the function is

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

where a_o sets the zero for the function; if it is symmetric around the x axis then $a_o = 0$. Fourier series allowed us to write general solutions of oscillations and wave equations in terms of a sum of sines and cosines, thus (in part) justifying our focus on these sorts of solutions.

2. FIELDS

Fields are physical and take values at any point in spacetime (x, y, z, t) . The subject of fields goes beyond the "contact interaction" picture of Newtonian mechanics. Fields accurately model much of known physics. Our typical examples are gravitational, electric and magnetic fields. Fields are visualized with **field lines**. These semi-quantitative representations of the field

- are tangent to the field vector and show the direction of the field with arrows
- are closer together where the field is more intense and further apart where the field is weaker
- are drawn so that the relative strength of the changes is represented in the number of field lines.

It is best to use a full page to draw your field diagrams.

2.1. Gravitational. (Newtonian) gravitational fields are sourced by masses. The local gravitational field is "little g" which is $\mathbf{g} = \mathbf{F}_g/m$. By Newton's law of gravitation

$$\mathbf{g} = \frac{GM}{r^2} \hat{\mathbf{r}}$$

for a single source "charge" M . For more complicated configurations using the potential to find the field is usually the best option.

2.2. Electro- and Magneto-statics. Electric fields are produced from electric charges. The electric field of a point charge Q is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o} \frac{Q}{r^2} \hat{\mathbf{r}}$$

Notice that this equations and $\mathbf{F} = q\mathbf{E}$ give Coulomb's relation: the force between any two point charges is proportional to the product of the charges and $1/r^2$. The parallels with the gravitational case are striking.

Two opposite charges at a fixed distance d apart form a dipole with moment qd . Immersed in an electric field the dipole enjoys a torque $\vec{\tau} = \mathbf{p} \times \mathbf{E}$.

The electric potential V in ("volts") is defined relative to your favorite reference point (e.g. tip of the whale's tail or $r = \infty$). It is the amount of work required to move a unit of charge from the reference point to where the potential is evaluated.

$$\Delta V = V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{r}. \text{ So for a single charge } V(r) = \frac{1}{4\pi\epsilon_o} \frac{Q}{r}$$

which gives us the easiest method of finding electric fields - add up all the electric potentials.

Magnetic fields are sourced by *moving* charges, such as current flow I . From Biot-Savart's Law,

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int \frac{Id\vec{\ell} \times \hat{\mathbf{r}}}{r^2}$$

Motion of a charged particle (q) can be found using

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (11)$$

As it turns out - we didn't study equations of motion leading to this result - E&M fields propagate as waves. For waves in the x -direction the fields satisfy

$$c^2 \frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(similarly for $\mathbf{B}$) in vacuum, where $c = \sqrt{1/\epsilon_o\mu_o}$, the speed of light, about 3×10^8 m/s.

3. WAVES

Familiar from observations of water the examples we studied this semester were string, spring, light, sound, gravity, and water waves. The *wave equation* for $u(x, t)$ in one spatial dimension is

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \quad (12)$$

It has solutions $u(x, t) = f(kx \pm \omega t)$ where f is any function and the *phase velocity* $v = \omega/k$. We most often consider periodic solutions for instance the right moving wave

$$u(x, t) = u_o \sin(kx - \omega t)$$

Energy transport for a wave is quantified by intensity the "energy flow per unit area". For waves on a string $I = 2\pi^2 \mu v f^2 A^2$. For light and sound we used the proportionality $I \propto |A|^2$ to derive several of our results, e.g. polarization and interference patterns.

Waves are characterized by their frequency (color), polarization (if transverse) and intensity. Waves satisfy the **superposition principle** (since the equation of motion is linear). This gives rise to many effects including interference, standing waves, and diffraction. We found that just thinking about the relative phase of sources was a useful starting point

$$\frac{\Delta L}{\lambda} = \frac{\Delta \varphi}{2\pi} \quad (13)$$

where ΔL is the path difference and $\Delta \varphi$ is the path change due to that difference.

3.1. Doppler Shifts. If the source and/or observer is moving then we have the ‘Doppler effect’, which is a shift in frequency. For *sound* we have

$$f' = f \left(\frac{v \pm v_{obs}}{v \mp v_{source}} \right). \quad (14)$$

For *light* the relation can only depend on the relative velocity and so there is a simpler relation

$$f' = f \sqrt{\frac{c \pm v}{c \mp v}}. \quad (15)$$

3.2. Geometris Optics. The “Waves travel in straight lines” approximation. It is good when the devices uses (such as slits) are large compared to the wavelength. The straight lines are rays.

Snell’s relation

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (16)$$

3.3. Physical Optics. “Waves are wavy”: Based on Huygens’ principle, the main effects are *interference* and *diffraction*. For instance, for Young’s double slit experiment interference between the two sources gives maximum amplitude at

$$d \sin \theta = n\lambda \quad (17)$$

in which d is the slit spacing and $n = 0, \pm 1, \pm 2, \dots$. Such effects are seen when the size of the devices used, e.g. slits, lenses, are within an order of magnitude of the wavelength of the wave. Physical optics includes geometric optics as a limit.

A single slit produces a *diffraction* pattern. The location of the dark bands are determined by

$$a \sin \theta = m\lambda \quad (18)$$

where a is the single slit width and $m = 1, 2, 3, \dots$ **Phasors** provide a way to find the intensity. We use them to find the amplitude, then we square them. They incorporate both interference and diffraction. For example we found that the double slit intensity is

$$I = I_o \left[\frac{\sin \left(\frac{\pi a \sin \theta}{\lambda} \right)}{\frac{\pi a \sin \theta}{\lambda}} \right]^2 \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right) \quad (19)$$

3.4. Wave resonance. Confined to an interval, waves exhibit resonance (e.g. standing waves) when the interval allows constructive interference between incident and reflected waves.

4. EXPERIMENTAL METHODS

There are many techniques you learned in lab, e.g. How to accurately time the period of pendulum? Start and stop the interval as it swings through the center! And time many periods to reduce uncertainty! How to avoid alignment errors? Gather data on both sides! etc.

When taking measurements there are:

4.1. Significant figures. “Sig figs” allow us to track the precision of our quantitative work. Give too many and you are including digits that have no meaning.

4.2. Uncertainties. : To propagate uncertainties first look for dominant error and then add all *independent* uncertainties in quadrature. For addition and subtraction

$$z = x + y \text{ or } z = x - y \text{ then } \delta z = \sqrt{\delta x^2 + \delta y^2} \quad (20)$$

For multiplication and division then add the relative uncertainties in quadrature,

$$z = xy \text{ or } z = x/y \text{ then } \frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2} \quad (21)$$

For a power, multiply the relative uncertainty by the power, i.e. if

$$z = x^n \text{ then } \frac{\delta z}{z} = n \frac{\delta x}{x}. \quad (22)$$

In general for a calculated quantity $q = q(x, \dots, z)$ then

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x} \delta x\right)^2 + \dots + \left(\frac{\partial q}{\partial z} \delta z\right)^2} \quad (23)$$

We also used the weighted average to combine results with significantly varying uncertainties.