Quantum (PHYS 290): Solutions 3

Fall 2025 v1.2

Solutions:

(1) Probability amplitudes and probabilities

(a) The amplitude is

$$z = \frac{1}{1+i} = \frac{1-i}{(1-i)(1+i)} = \frac{1-i}{2}.$$

(It is best to work with $x = iy$ or $re^{i\phi}$ forms.) So the probability

$$P = |z|^2 = z^* z = \frac{(1+i)}{2}\frac{(1-i)}{2} = \frac{1}{2}$$

(b) $z = i$ so

$$P = |z|^2 = z^* z = (-i)(i) = 1$$

(c) Now

$$z = \frac{1}{1+i} + i = \frac{1-i}{2} + i = \frac{1+i}{2}$$

So the probability is

$$P = |z|^2 = z^* z = \frac{(1+i)}{2}\frac{(1-i)}{2} = \frac{1}{2}.$$

Same as (a)

(d) But

$$z = \frac{1}{1+i} - i = \frac{1-3i}{2}$$

So the probability is

$$P = |z|^2 = z^* z = \frac{(1+3i)}{2}\frac{(1-3i)}{2} = \frac{10}{4} > 1$$

so this cannot be a probability and z cannot be a probability amplitude.

(2) There is a band of polarized light spanning the sky at a 90 degree angle from the direction of sunlight - where the sky is 'bluest'. I found this by looking through and rotating the polarizer in different parts of the sky.

The light from this part of the sky is polarized perpendicular to the sun's rays. This starts to make sense if you recall that light is a transverse wave and if we think of a model of little oscillators set in motion by the wave. It turns out that the particles oscillating along the line of sight to do emit radiation (light) towards us.

BTW the details of this scattering are often discussed in Phys 480, Electrodynamics. Ask for Chapter 11!

(3) Finding amplitudes and probabilities for the Michelson interferometer

(a) Accounting for reflections (2 of them, one off the mirror and one off the beam splitter) and the accumulated phase over path 1 and distance d_1

$$z_1 = e^{ikd_1} e^{i\pi} e^{i\pi} = e^{ikd_1}$$

The second path has one reflection and phasor

$$z_2 = e^{ikd_2} e^{i\pi} = -e^{ikd_2}$$

We asked to express this in terms of the arm lengths. Each path travels up and down the arms so we have $2l$ in each distance. Letting the rest of the phase be $e^{i\alpha}$, I have

$$z_1 = e^{ikd_1} = e^{i\alpha}e^{ik2l_1} \text{ and } z_2 = -e^{ikd_2} = -e^{i\alpha}e^{ik2l_2}.$$

The total amplitude is in the sum

$$\begin{aligned} z = z_1 + z_2 &= e^{i\alpha}\left(e^{ik2l_1} - e^{ik2l_2}\right) \\ &= e^{i\alpha}e^{ik2l_2}\left(e^{ik2(l_1-l_2)} - 1\right) = 2ie^{i\alpha}e^{ik(l_1+l_2)}\sin(k(l_1-l_2)) \end{aligned}$$

Hence the probability is $\sin^2(k(l_1-l_2))$ or

$$P = \sin^2\left(\frac{2\pi}{\lambda}(l_1-l_2)\right)$$

(b) For a probability of 1,

$$\sin^2(k(l_1-l_2)) = 1$$

so

$$\frac{2\pi(l_1-l_2))}{\lambda} = \pi\frac{(\text{odd integer})}{2} \implies \Delta l = l_1 - l_2 = \frac{(\text{odd integer})\,\lambda}{4}.$$

(c) Choosing the odd integer to be 1 the additional length of $\lambda/6$ gives

$$\Delta l = \frac{\lambda}{4} + \frac{\lambda}{6} = \frac{5\lambda}{12}$$

so

$$P = \sin^2\left(\frac{2\pi}{\lambda}\Delta l\right) = \sin^2(5\pi/6) = 1/4.$$

This result also holds for other odd integers.

(4) Single slit diffraction via integration: Townsend gives us the amplitudes that we need to integrate over the slit from the top ($x=0$) to the bottom ($x=a$), as shown in below (and Figure 1.44). Here's the integral

$$z_P = \int dz_P = \frac{r}{a}e^{ikd_1}\int_0^a e^{ik\sin\theta\, x}dx.$$

Lots of ways to do this but my favorite is to do a change of variables. Let

$$\begin{aligned} &u = ik\sin\theta\, x \text{ so that } du = ik\sin\theta\, dx. \\ &\text{On the boundary, } u = 0 \text{ when } x = 0, \text{ and} \\ &u = ika\sin\theta \text{ when } x = a. \end{aligned} \tag{1}$$

Then

$$z_P = \frac{r}{a}e^{ikd_1}\frac{1}{ik\sin\theta}\int_0^{ika\sin\theta} e^u du = \frac{r}{a}e^{ikd_1}\frac{1}{ik\sin\theta}\left(e^{ika\sin\theta} - 1\right).$$

To save some algebra it is handy to re-write this as a sine, pulling out a factor of $e^{ika\sin\theta/2}$,

$$z_P = \frac{r}{a}e^{ikd_1}\frac{1}{ik\sin(\theta)}e^{ika\sin(\theta)/2}\left(e^{ika\sin(\theta)/2} - e^{-ika\sin(\theta)/2}\right) = re^{ikd_1}e^{ika\sin(\theta)/2}\,\frac{\sin[ka\sin(\theta)/2]}{ka\sin(\theta)/2}$$

Therefore,

$$z_P^* z_P = r^2\left\{\frac{\sin[ka\sin(\theta)/2]}{ka\sin(\theta)/2}\right\}^2$$

as expected.

The minimum for the single slit pattern occurs when $z_P = 0$. This happens when $\sin[ka\sin(\theta)/2] = 0$ or when

$$\frac{2\pi}{\lambda}a\frac{1}{2}\sin\theta = \frac{\pi a\sin\theta}{\lambda} = n\pi \text{ with } n = \pm1, \pm2, \ldots$$

The last equality is equivalent to the angular locations of the minima of single slit diffraction given in equation (1.17).

If $ka \ll 1$ and the slit width a is much less that a wavelength the argument of the sine function is small, that is

$$\sin\left[ka\sin(\theta)/2\right] \simeq ka\sin(\theta)/2.$$

This means that

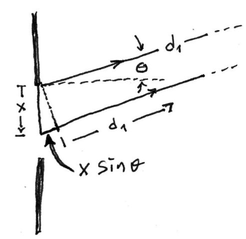
$$z_P \propto \frac{[ka\sin(\theta)/2]}{ka\sin(\theta)/2} = 1$$

so in this limit the probability $z_P^* z_P$ should be flat. We indeed can see this behavior since in Figure 1.9 the top panel is approximately constant while the bottom panel in varies significantly over the same angular scale.

BTW a key point about the setup for this problem is that the distance from the top of the slit to the point P is d_1. This means that the accumulated phase of the phasor at the top of the slit is e^{ikd_1}. This basically sets the reference phase for all the other phasors. Just as in the usual slit case the additional phase due to the change in path length is

$$\Delta\varphi(x) = kx\sin\theta$$

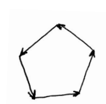
where the phase change depends on position x. We can se this in the sketch



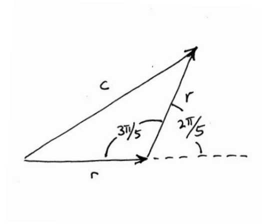
The is the relative phase for a source at x. Now we are going to integrate up these sources and purely on dimensional grounds we must have an inverse factor of length in the wee contribution dz_P. The only dimension of the apparatus in the problem is the slit width is a so

$$dz_P(z) = re^{ikd_1}\frac{1}{a}e^{i\Delta\varphi(x)}dx = re^{ikd_1}\frac{r}{a}e^{ikx\sin\theta}$$

(5) Following the suggestion let's think about that first minimum. It occurs when the phasors "wrap around" to add to sum to zero for the first time when moving away from the central maximum. Keeping in mind that each phasor is related to the previous one by the same angle, we have



Ok, if we now block the last three slits then these phasors go away. The picture is now



where I have added the phasors' lengths and the 5 slit relative phase of $2\pi/5 = 72°$. The resultant phasor has length c. Using law of cosines,

$$c^2 = 2r^2 - 2r^2\cos(3\pi/5) \simeq 2.6r^2 \implies c \simeq 1.62\, r$$

Hence relative to r the probability is $c^2 \simeq 2.6\, r^2$. (Determining the absolute probability looks problematic so I'll leave it at that.)

(6) A photon with a wavelength equal to the Compton wavelength $\lambda_C = h/m_e c$ collides with a free electron at rest:

(a) At π cosine is -1 so the $1 - \cos\theta$ factor becomes 2. From last week's solutions

$$\frac{E_\gamma}{E_{\gamma'}} = 1 + \frac{E_\gamma}{m_e c^2}\left(1 - \cos\theta\right).$$

With $E_\gamma = hc/\lambda_C = m_e c^2$,

$$E_{\gamma'} = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2}\left(1 - \cos\theta\right)} \simeq \frac{E_\gamma}{1 + \frac{2E_\gamma}{m_e c^2}} = \frac{1}{3} m_e c^2$$

(b) The rest of the energy must be in the KE of the electron, $2/3\, m_e c^2$.

(7) Interference with C_{60}! For this multiple slit interference the slits were 50 nm wide and d =100 nm apart. The "screen" was L =1.25 m away.

(a) The 117 m/s C_{60}'s had a mass of $m_{C_{60}} \simeq 1.2 \times 10^{-24}$ kg so the de Broglie wavelength was

$$\lambda = \frac{h}{p} = \frac{h}{mv} \simeq 4.7 \times 10^{-12}\ \text{m}.$$

(b) The expected spacing is $d \sin\theta = n\lambda$. The spacing Δy in the small angle limit is then

$$\Delta y = \frac{\lambda L}{d} \simeq 59 \mu m.$$

Averaging over 5 fringes, which span about 180 μm, I find that the spacing should be about 45 μm. The agreement isn't spectacular but not horrible either.

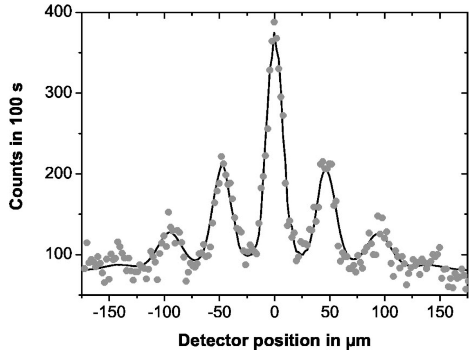


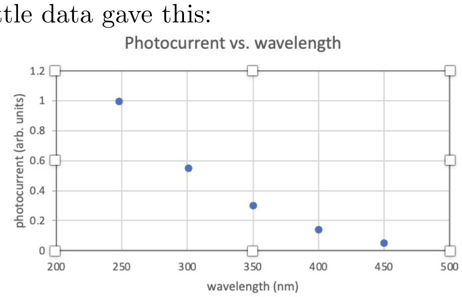
Fig. 7. Far-field diffraction of C_{60} using the slotted disk velocity selector. The mean velocity was $\bar{v} = 117$ m/s, and the width was $\Delta v/v \sim 17\%$. Full circles represent the experimental data. The full line is a numerical model based on Kirchhoff–Fresnel diffraction theory. The van der Waals interaction between the molecule and the grating wall is taken into account in form of a reduced slit width. Grating defects (holes) additionally contribute to the zeroth order.

(8) Based on the PhET photoelectric simulation.

(a) Electrons are freed! They appear to have a variety of speeds, since they move across the gap at different rates. But the current is a constant 0.071 A. (All with target material set to "Sodium", the lightbulb set to 400 nm, volts to 0 V, and the light intensity set to 50%.)

(b) With the light intensity at 100% the picture looks pretty similar but he current went up to 0.141 A, i.e. it doubled.

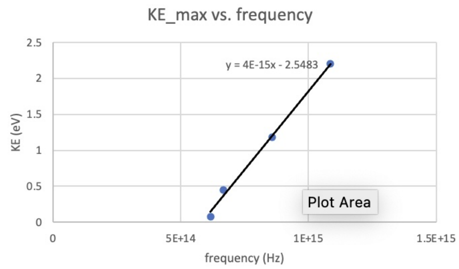
(c) Shorter wavelength process more energetic (fast) electrons. The current also varies with wavelength. Taking a little data gave this:



No electrons are emitted for wavelengths greater than ~ 540 nm.

(d) At 450 nm and 100% intensity, the voltage that "just barely stops" the electrons from reaching the righthand plate is $V = -0.45$ V. At 350 nm this stopping potential is much higher (more negative) $V = -1.18$ V.

(e) I took a little data for the stopping potential vs. frequency. Due to energy conservation the maximum kinetic energy in units of eV is the stopping potential ($KE = eV_{stop}$). So I plotted the data as in Figure 1.14 in Townsend:



(f) By energy conservation the frequency of the incoming photon f is related to the work function W and stopping potential eV_{stop} as

$$hf = KE + W = eV_{stop} + W \implies eV_{stop} = hf - W.$$

So the y-intercept is the (minus) the work function and the slope is Planck's constant h. From the above plot it looks good since $h \sim 4.1 \times 10^{-15}$ eV s and $W \sim 2$ (from Figure 1.14).