Quantum (PHYS 290): Solutions 4

Fall 2025 v1.0

Solutions:

(1) From the de Broglie wavelength we can find the speed via

$$v = \frac{p}{m} = \frac{h}{\lambda\, m_{He}} \simeq 970 \text{ m/s},$$

with the given wavelength and the mass of He, $m_{He} = 6.63 \times 10^{-27}$ kg.

(2) He interference with uncertainties: For full credit solutions must have uncertainty in y.

(a) As above,

$$v = \frac{p}{m} = \frac{h}{\lambda\, m_{He}} \simeq 1780 \simeq 1800 \text{ m/s},$$

(b) The spacing between fringes, y, in double slit interference at small angles $\theta \simeq y/L$ gives

$$d \sin\theta \simeq \frac{y\, d}{L} = \lambda \implies y = \frac{\lambda L}{d} \simeq 4.48\ \mu\text{m}.$$

The uncertainty is

$$\delta y = \frac{L\lambda}{d^2}\delta d = y\frac{\delta d}{d} \simeq 0.3\ \mu\text{m}$$

so the final result is

$$y = 4.5 \pm 0.3\ \mu\text{m}$$

which is in excellent agreement with the stated result.

(3) The minima for single slit diffraction occur at

$$a \sin\theta = m\lambda.$$

The distance, let's call it y_a, to the first minima ($m = \pm 1$) is

$$y_a = \frac{\lambda L}{a}$$

using the small angle approximation. This relation is *very similar* to the last problem but this one is for the position of the first minimum of the finite width diffraction pattern. The question asks for the total distance between the minima or $2y_a \simeq 170\,\mu$m, which looks roughly correct if the diffraction minimum are on the far left and right (in the noisy bits just before the data drops to 0) in figure 2.4(b).

Using both $d \sin\theta = n\lambda$ and the small angle approximation, the interference fringes are located at about

$$y_d = \frac{\lambda L}{d} \simeq 11\ \mu\text{m}$$

apart. Dividing the width by the spacing $2y_a/y_d = 16$. However, the diffraction minima cancel the bright fringes on either side so that means only the $n_{max} = 7$ (7th order) fringes are visible. Since the central maximum is $n = 0$ this means that the total number is $2n_{max}+1 = 15$ fringes.

The parameters in this problem are discussed in section 2.1. I gleaned $a = 1\,\mu$m, $d = 8\,\mu$m, and $L = 195$ m for this He experiment.

[Oddly in the diagram I count 9 fringes between these minima. Nine fringes amount to a total distance on the observing plane of about 100 μm. The agreement is not great (100 vs 170).]

(4) The Planck length ℓ_P: I use square brackets for "dimensions of" and L for length T for time and M for mass. You can also so this in SI units for instance. For full credit the solution must derive the (unique) relation for ℓ_P.

(a) The speed of light has dimensions

$$[c] = \frac{L}{T}.$$

From Newton's law of gravity,

$$F = -\frac{GmM}{r^2} \text{ or, dimensionally, } \frac{ML}{T^2} = \frac{[G]M^2}{L^2},$$

we can find the dimensions of Newton's gravitational constant,

$$[G] = \frac{L^3}{MT^2}.$$

From Einstein's energy relation

$$E = \hbar\omega,$$

we can find the dimensions of $\hbar$,

$$[\hbar] = \frac{ML^2}{T}.$$

With these dimensions we can find the equation for length. [This is essential part of the derivation.] Raising each constant to some power I find

$$L = [G]^a[\hbar]^b c^d \text{ gives } L = \left(\frac{L^3}{MT^2}\right)^a \left(\frac{ML^2}{T}\right)^b \left(\frac{L}{T}\right)^d.$$

This equation will give the formula for the Planck length. For instance looking at mass M, we see that $a = b$, for time $2a + b + d = 0$ and for length $1 = 3a + 2b + d$. Solving this set of equations gives $a = b = 1/2$ and $d = -3/2$ so

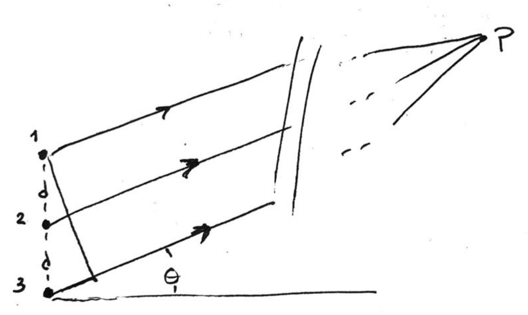
$$\ell_P = \sqrt{\frac{G\hbar}{c^3}}.$$

(b) This works out to about 1.6×10^{-35} m. Tiny!

(c) For a 0.05 kg tennis ball the speed is

$$v = \frac{h}{m\,\ell_P} \simeq 820 \text{ m/s}.$$

(5) The phasor work for a three slit pattern. Here's the set-up



I'll call the phasor from source 1, at the top,

$$z_1 = re^{ikx}$$

where x is the distance from source 1 to the distant point P. The path from source 2 is $d\sin\theta$ longer so

$$z_2 = re^{i(kx+kd\sin\theta)} = re^{i(kx+\phi)},$$

in the last step using the definition of ϕ in the problem. The last slit has a phasor

$$z_3 = re^{i(kx+2kd\sin\theta)} = re^{i(kx+2\phi)}.$$

Each of these contribute to the total phasor at P, z_P

$$z_P = re^{ikx}\left(1 + e^{i\phi} + e^{i2\phi}\right).$$

The algebra is faster if we note

$$z_P = re^{ikx}e^{i\phi}\left(e^{-i\phi} + 1 + e^{i\phi}\right) = re^{i(kx+\phi)}\left(1 + 2\cos(\phi)\right).$$

The magnitude squared is then

$$P = |z_P|^2 = r^2\left[1 + 4\cos(\phi) + 4\cos^2(\phi)\right].$$

Alternatively, the magnitude squared is

$$z_P^* z_P = r^2\left(1 + e^{-i\phi} + e^{-i2\phi}\right)\left(1 + e^{i\phi} + e^{i2\phi}\right).$$

Expanding the many terms and gathering them gives

$$z_P^* z_P = r^2\left(3 + 2e^{i\phi} + 2e^{-i\phi} + e^{i2\phi} + e^{-i2\phi}\right) = r^2\left[3 + 4\cos(\phi) + 2\cos(2\phi)\right]$$

which, using the double angle formula $\cos 2\phi = 2\cos^2\phi - 1$, gives the desired result.

(6) If $\Psi(x,t) = c_1\psi_1(x,t) + c_2\psi_2(x,t)$, where ψ_i are solutions to the Schrödinger equation, then with the derivative is

$$\frac{\partial^2\Psi}{\partial x^2} = c_1\frac{\partial^2\psi_1}{\partial x^2} + c_2\frac{\partial^2\psi_2}{\partial x^2}.$$

Similarly,

$$\frac{\partial\Psi}{\partial t} = c_1\frac{\partial\psi_1}{\partial t} + c_2\frac{\partial\psi_2}{\partial t}.$$

So Schrödinger equation with Ψ,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

becomes

$$-c_1\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2} - c_2\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} + V(c_1\psi_1 + C_2\psi_2) = i\hbar\left(c_1\frac{\partial\psi_1}{\partial t} + c_2\frac{\partial\psi_2}{\partial t}\right)$$

But this is just c_1 times

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2} + V\psi_1 = i\hbar\frac{\partial\psi_1}{\partial t},$$

plus c_2 times

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} + V\psi_2 = i\hbar\frac{\partial\psi_2}{\partial t}.$$

But both these are satisfied so Ψ is also a solution.

(7) Thinking about the probability density $|\psi|^2$ of ψ in the plot shows that the highest probability density is around $x = -1$ since the $|\psi|^2$ is a relatively large 1 there. There is a chance to measure the particle around 2 but this probability density is much lower, at roughly 0.04. The particle won't be found around 0, less than -2, and greater than 4 because $\psi = 0$ in these regions. The particle is more likely to be found in position $x < 0$ than $x > 0$ since that probability - the area under the $|\psi|^2$ curve - there is larger.

(8) 'Adding' in the time dependence is just multiplying by the phase for that energy. Here,

$$\Psi(x,t) = \sqrt{\frac{2}{3}}\psi_1 e^{-iE_1t/\hbar} + \sqrt{\frac{1}{3}}\psi_2 e^{-iE_2t/\hbar}.$$

(In class we used "u" instead of "ψ".)The probability of measuring E_1 is the amplitude squared of the first wavefunction or 2/3.

The energy expectation value is

$$\begin{aligned}\langle E\rangle &= \sum E_i P(i) = \frac{2}{3}E_1 + \frac{1}{3}E_2 \\ &= \left(\frac{\hbar^2\pi^2}{2mL^2}\right)\left(\frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 2^2\right) \\ &= \frac{\hbar^2\pi^2}{mL^2}\end{aligned}$$

Measure these? There are lots of ways... Drop the potential to zero and measure the speed the particle travels. Measure transitions between states by sending in and receiving light in the form $E = \hbar\omega$. Couple this to another system to make a superposition of states of two particles and make measurements on the entangled particle....

(9) The probability that a particle will be found in the interval $0 \le x \le L/4$ when it is in the $n = 2$ state is

$$P(0 \le x \le L/4) = \int_0^{L/4} u_2^2(x)dx = \frac{2}{L}\int_0^{L/4}\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{2}{L}\cdot\frac{L}{8} = \frac{1}{4}$$

I used the handy relation that the integral over sine squared is 1/2 the length of the interval.

This is not too surprising since u_2 is a delocalized energy eigenstate; it has equal probability to be found anywhere in the well.

(10) (Optional extra 1 pt.: Seth will grade these)