Solutions:

(1) First the velocities. The given phase velocity is

$$v_{ph} = \sqrt{\frac{kT}{\rho}} \text{ which is } \frac{\omega}{k}$$

by definition. So this means

$$\omega = \omega(k) = \sqrt{\frac{k^3 T}{\rho}}.$$

The group velocity is then

$$v_g = \frac{\partial \omega}{\partial k} = \frac{3}{2}\sqrt{\frac{kT}{\rho}} = \frac{3}{2} v_{ph}.$$

So this means that the group or circular bump moves out with $v_g = 1.5 v_{ph}$, *faster* than the ripples that are superimposed to make the bump. Therefore it would seem that the little ripples are passed by the bump.

(2) (2 pts.) de Broglie's relativistic adventure

(a) Since

$$E^2 = p^2 c^2 + m^2 c^4$$

then using $E = \hbar\omega$ and $p = \hbar k$,

$$\hbar\omega = \sqrt{(\hbar k)^2 c^2 + m^2 c^4} \text{ or } \omega = \sqrt{(kc)^2 + (mc^2/\hbar)^2}.$$

(b) Taking the derivative gives the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{k\hbar c^2}{\sqrt{(\hbar k c)^2 + m^2 c^4}}.$$

The rest of this part of the problem allows us to check this relation in the relativistic limit. Pulling out a factor of $\hbar k c$ out of the square root,

$$v_g = \frac{k\hbar c^2}{\hbar k c\sqrt{1 + [mc^2/(\hbar k c)]^2}}.$$

gives for large p (or k)

$$v_g \simeq c\left[1 - \frac{1}{2}\left(\frac{mc}{\hbar k}\right)^2\right].$$

where I have used the handy relation $(1+x)^n \simeq 1 + nx$ for $x < 1$.
Townsend gives us de Broglie's original mass limit so we can check this. He quotes

$$\frac{c - v_g}{c} = 0.01$$

when $v_g = 0.99c$. Looking back at the expression of v_g in the limit we see that this expression is tailored to give us the mass term.

$$\frac{c - v_g}{c} = \frac{1}{2}\left(\frac{mc}{\hbar k}\right)^2$$

Expressing this in terms of wavelength gives

$$\frac{c - v_g}{c} = \frac{1}{2}\left(\frac{\lambda mc}{h}\right)^2$$

so that the photon mass is

$$m_\gamma = \frac{\sqrt{0.02}\,h}{\lambda c} \simeq 1.0 \times 10^{-47}$$

as Townsend quotes.

(c) Re-running the numbers for 1 m radio waves and

$$\frac{c - v_g}{c} < 4 \times 10^{-7}$$

gives

$$m_\gamma < 2 \times 10^{-45} \text{ kg},$$

this time with actual data.

(3) Uncertainties! Computing the expectation value of x

$$\langle x \rangle = \langle \psi \mid x \mid \psi \rangle = \int \psi^*(x)\, x\, \psi(x) dx = \frac{2}{L}\int_0^L x \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{L}{2}$$

and x^2

$$\left\langle x^2 \right\rangle = \langle \psi \mid x^2 \mid \psi \rangle = \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{L^2}{6}\left(2 - \frac{3}{\pi^2}\right).$$

I used Mathematica for the integration. The uncertainty in x is

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{L}{\sqrt{2}\pi}\left(\frac{\pi^2}{6} - 1\right)^{1/2}.$$

As for the momentum, it is a derivative

$$\langle p \rangle = \langle \psi \mid \hat{p} \mid \psi \rangle = -i\hbar \int \psi^*(x)\, \frac{d}{dx}\, \psi(x) dx = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) dx = 0$$

and $\left\langle p^2 \right\rangle$

$$\left\langle p^2 \right\rangle = \langle \psi \mid \hat{p}^2 \mid \psi \rangle = -\hbar^2 \int \psi^*(x)\, \frac{d^2}{dx^2}\, \psi(x) dx = \frac{2\pi^2\hbar^2}{L^3}\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx = \left(\frac{\hbar\pi}{L}\right)^2.$$

These results give the uncertainty

$$\Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar\pi}{L}.$$

Finally, the product is

$$\Delta x\, \Delta p = \frac{\hbar}{\sqrt{2}}\left(\frac{\pi^2}{6} - 1\right)^{1/2} \simeq 0.56\hbar \geq \frac{\hbar}{2}$$

as expected from the Heisenberg uncertainty relation. From this we see that the $n = 1$ energy-eigenstate state is not a minimum uncertainty state.

(4) Sorry! There was some confusion about the statement of this problem in office hours which I didn't notice until late in the afternoon. I'll first give the solution to the problem in the book and then discuss the confusion.

(a) If the well suddenly expands to twice the width with *only the right wall* moving then the $(n = 1)$ wavefunction,

$$u_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \text{ on } 0 < x < L$$

and 0 outside this interval, can now occupy $L < x < 2L$, too. We can find the new well's ground state wavefunction by substituting $2L$ for L. The new wavefunction $\tilde{u}_1$ is

$$\tilde{u}_1(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{2L}\right) \text{ on } 0 < x < 2L.$$

The original wavefunction $u_1(x)$ can be expanded in the basis of the wavefunctions in the new well,

$$u_1(x) = \sum_n c_n \tilde{u}_n(x).$$

To answer the question of "how much of u_1 is in $\tilde{u}_1$?" we can compute the inner product (or "overlap")

$$c_1 = \int_0^{2L} u_1\, \tilde{u}_1 dx = \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{2L}\right) dx$$

Note the key point that the original wavefunction vanishes on the right side of the new well so the upper limit on integration goes from $2L$ to L. Evaluating the integral in Mathematica I find

$$c_1 = \frac{4\sqrt{2}}{3\pi}.$$

Squaring this gives about 0.36, the probability of u_1 transitioning to the ground state in the new well.
Similarly for the probability of transition to the first excited state $(n = 2)$

$$c_2 = \int_0^{2L} u_1\, \tilde{u}_2 dx = \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{2L}\right) dx = \frac{1}{\sqrt{2}}.$$

Hence the probability is $1/2$. (The $n = 3$ transition probability is about 0.12 and so on.)

(b) The evolution of the new state is

$$\Psi_(x,t) = \sum_n c_n \tilde{u}_n(x) e^{iE_n t/\hbar},$$

with the c_n's as above and the E_n's are the energies in the new well, $E_n = n^2\pi^2\hbar^2/8mL^2$. Since these evolve at different relative phases Ψ is not a stationary state.

Now the confusion: In office hours we talked about the problem when *both walls* move outward by the same amount. This change respects the symmetry of the wavefunction and so it just transitions to the ground state of the new well. In this case $c_2 = 0$ and the new state is a stationary state.

Graders please accept either solution.

(5) The asymmetric solutions for a particle in a finite square well are sines. Picking up the work in class just after the application of the boundary conditions at $a/2$ - we divided them to find

$$\frac{1}{\kappa} = \frac{Ae^{-ika/2} + Be^{ika/2}}{ik\left(Ae^{-ika/2} - Be^{ika/2}\right)}.$$

Equivalently see page 115 and equation (4.16).

(a) With $B = -A$ (the source of the antisymmetry) the above equation gives

$$\frac{ik}{\kappa} = \frac{e^{-ika/2} - e^{ika/2}}{e^{-ika/2} + e^{ika/2}} = -i\tan(ka/2).$$

Simplifying this gives

$$-\cot(ka/2) = \frac{\kappa a/2}{ka/2},$$

where I have added the factors for the next step, the change of variables. This is the quantization condition. To solve this it is useful to let

$$\xi = ka/2 = \frac{a}{\hbar}\sqrt{\frac{mE}{2}} \tag{1}$$

and

$$\xi_o = \frac{a}{\hbar}\sqrt{\frac{mV_o}{2}}.$$

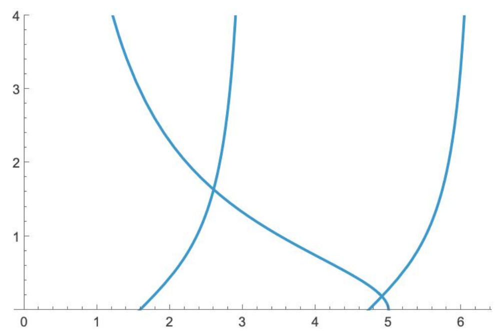
Then

$$\kappa = \frac{\sqrt{2m(V_o - E)}}{\hbar} = \sqrt{\xi_o^2 - \xi^2}$$

so the quantization condition becomes

$$-\cot(ka/2) = \frac{\sqrt{\xi_o^2 - \xi^2}}{\xi}.$$

(b) To solve the quantization condition when $\xi_o^2 = 25$ we can graph the left and right hand sides. The result is



The intersection points are the solutions. I used the FindRoot function in Mathematica to find the two solutions

$$\xi_1 \simeq 2.596 \text{ and } \xi_2 \simeq 4.906.$$

To find the energies we can solve for E in equation 1 giving

$$E_n = \frac{2\xi_n^2\hbar^2}{ma^2}.$$

So,

$$E_1 \simeq \frac{13.5\hbar^2}{ma^2} \text{ and } E_2 \simeq \frac{48.1\hbar^2}{ma^2}$$

(6) This is a qualitative or semi-quantitative problem. In figure 4.4 there is a trend of increasing amplitude and extent of the wavefunction in the region $x > a/2$ which suggests a higher probability. The probability of finding the particel outside the well is twice the integral of $|\psi|^2$ over the region $a/2 < x < \infty$. So more non-vanishing ψ the greater the probability. That's the basic idea.

Can we do a bit better? Townsend asks that we look at the wavefunction in this region. It is

$$\psi \sim e^{-\kappa_n x} \text{ where } \kappa_n = \frac{\sqrt{2m(V_o - E_n)}}{\hbar}$$

so as the energy E_n increases toward V_o, κ decreases and the decay is more gradual. For the moment neglecting the normalization, the probability of being outside for state n is proportional to

$$P_n \propto 2\int_{a/2}^{\infty} e^{-2\kappa_n x} = \frac{1}{2\kappa_n}\left[-e^{-2\kappa_n x}\right]_{a/2}^{\infty} = \frac{e^{-\kappa_n a}}{2\kappa_n}.$$

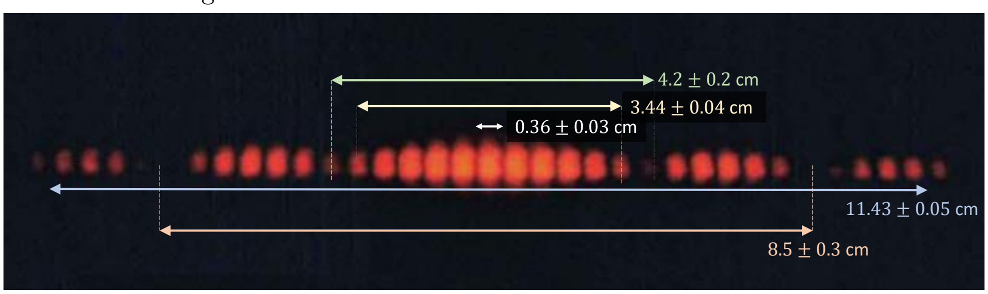
Taking a ruler to figure 4.4(a) I find that the $E_4 \simeq 0.96V_o$ and $E_1 \simeq 0.06V_o$. Comparing the probabilities for $n = 4$ and $n = 1$ we see

$$\frac{P_4}{P_1} = \frac{e^{-\kappa_4 a}}{\kappa_4} \cdot \frac{\kappa_1}{e^{-\kappa_1 a}} \simeq \sqrt{\frac{0.94}{0.04}} e^{-(\kappa_4 - \kappa_1)a} \simeq 5e^{-(\kappa_4 - \kappa_1)a}$$

Thus, even neglecting the normalization and the exponential factor it is at least *at least* five times more likely to find the $n = 4$ state outside than the $n = 1$ state.

The precise solution requires the normalized solutions, which are considerably more involved.

(7) The useful measurements are all for double sided full widths so we'll need to take this into account. The wavelength is 662.0 nm and $L = 2.15$.



(a) The second order ($m = 2$) minima are clearly visible. (I'm not so confident in the measurement in the first order shown in green at the top.) So using the $\Delta y_2 = 8.5 \pm 0.3$ cm and

$$a \sin\theta = 2\lambda \text{ or with small angles } \frac{a\Delta y_2}{2L} = 2\lambda$$

The small angle approximation is probably fine (we can check it in a moment) due to the long distance $L = 2.15$ m. I added a factor of 1/2 since Δy_2 is the distance between second order minima, not measured from the center where $\theta = 0$. The width of the slits should then be

$$a = \frac{4L\lambda}{\Delta y_2} \simeq 6.698 \times 10^{-5} \text{ m}.$$

There's only uncertainty in Δy_2 and so

$$\frac{\delta a}{a} = \frac{\delta \Delta y_2}{\Delta y_2} = \frac{0.3}{8.5} \implies \delta a \simeq 2 \times 10^{-6} \text{ m}.$$

The result is then $a = 6.7 \pm 2 \times 10^{-5}$ m $= 67 \pm 2\,\mu$m. (Checking the angle $\Delta y_2/2L \simeq 0.02$, which is small.)

(b) Given the beautiful pattern I'd like a measurement out to a high order in the double slit interference pattern but it looks like the best we have is the fringes in the central diffraction max.[1] There are 11 bright fringes there. Sine $11 = 2n+1$ the last order is $n = 5$. The measurement is $\Delta x_5 = 3.44 \pm 0.04$ cm. Similarly to above

$$d \sin\theta = 5\lambda \text{ or with small angles } \frac{d\Delta x_5}{2L} = 5\lambda$$

The spacing of the slits should then be

$$d = \frac{10L\lambda}{\Delta x_5} \simeq 4.14 \times 10^{-4} \text{ m.}$$

There's only uncertainty in Δx_5 and so

$$\frac{\delta d}{d} = \frac{\delta \Delta x_5}{\Delta x_5} = \frac{0.04}{3.44} \implies \delta d \simeq 5 \times 10^{-6} \text{ m.}$$

The result is then $d = 4.14 \pm 0.05 \times 10^{-4}$ m $= 0.414 \pm 0.005\,$mm.

(8) I started out with a basic $y = x^2$ for 8 points. The script was (up to some formatting issues)

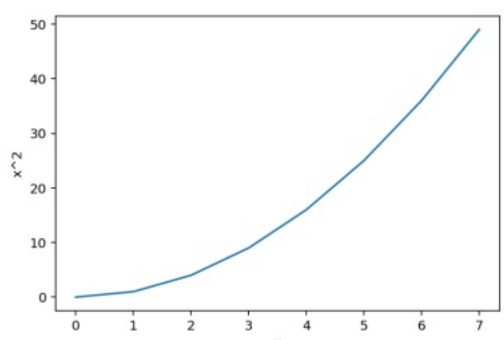
```
import numpy as np
import matplotlib.pyplot as plt

x=np.zeros(8)
y=np.zeros(8)

for i in range(0,8):
x[i]=i
y[i]=i*i

plt.plot(x,y)
plt.xlabel("x")
plt.ylabel("x^2")
plt.show()
```

giving the output



[1]You could use the high order double slit minima measurement but we haven't emphasized this.

Scatter plots often make sense so I switched to blue points (the weird ghostly "bo" in the plot command) and got fancy with the y label. Here's the revised script

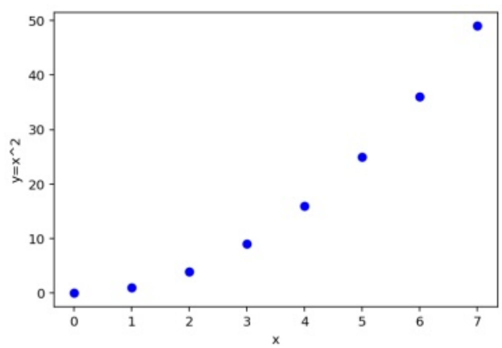
```
import numpy as np
import matplotlib.pyplot as plt

x=np.zeros(8)
y=np.zeros(8)

for i in range(0,8):
x[i]=i
y[i]=i*i

plt.plot(x,y,"bo")
plt.xlabel("x")
plt.ylabel("y=x^2")
plt.show()
```

and the plot



Your solution will likely differ from mine because of different number of points and/or different function as well as different methods in coding.