Quantum (PHYS 290): Guide 6 Solutions

Fall 2025 v1.0

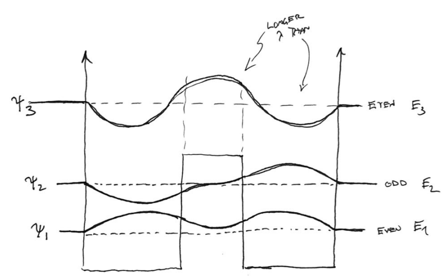
Solutions:

(1) Labeling the figures (i), (ii), (iii), and (iv) from top left, clockwise. (i) Acceptable. Looks like a decreasing potential with $E > V(x)$. (ii) Assuming trend continues ψ diverges so it is not an acceptable wavefunction. (iii) Not acceptable. It is not even a function! (iv) This is acceptable and in fact we have seen this in the finite square well.

(2) Things to be sure to include: (1) The wavefunction must vanish at the left and right hand sides, where the potential goes to ∞. (2) The wavefunction will wiggle when $E > V$. (3) The ground state should have "one" bump in a well. (4) Each level adds a bump. (5) Inside the barrier the TISE is

$$\frac{d^2\psi}{dx^2} - \kappa^2\psi = 0$$

so the wavefunction curves down when $\psi > 0$ and up when $\psi < 0$. My sketch looks like

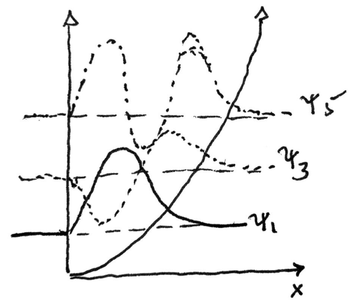


(3) There is a finite well below V_2 so for $0 < E < V_2$ the levels are discrete. For $V_2 < E < V_3$ the particle is not bounded so the allowed energies are continuous. We need boundary conditions on two sides to have discrete levels. There are no states for $E < 0$.

(4) We need wavefunctions that vanish at the origin. Looking at figure 4.13 we can spot the wavefunctions that do this and pass through the horizontal at $x = 0$. They are ψ_1, ψ_3, ψ_5, ...- the n is odd ones. This also looks good in the sense of counting bumps or maxima: ψ_1 has one in the new well, ψ_3 has two, etc. The three lowest energies are

$$E_1 = \frac{3\hbar\omega}{2},\ E_3 = \frac{7\hbar\omega}{2},\ \text{and}\ E_5 = \frac{11\hbar\omega}{2}.$$

I have kept the original labels, although one reasonably could number them other ways. Here's a sketch of the states



(5) (2 pts.) For $t \neq 0$ we multiply by the correct phases for each stationary state - the phase $e^{iE_n\,t/\hbar}$.

(a) Using $E_n = \hbar\omega(n+1/2)$ the time-dependent wavefunction of this superposition of $n=0$ and $n=1$ states is then

$$\Psi(x,t) = \frac{\sqrt{3}}{2}\psi_0(x)e^{-i\omega t/2} + \frac{1-i}{2\sqrt{2}}\psi_1(x)e^{-i3\omega t/2}.$$

(b) This wavefunction is a sum of normalized stationary states so the coefficients above should be the probability amplitudes. Let's check that they sum to one:

$$\left|\frac{\sqrt{3}}{2}\right|^2 + \left|\frac{1-i}{2\sqrt{2}}\right|^2 = \frac{3}{4} + \frac{(1-i)(1+i)}{8} = \frac{3}{4} + \frac{1}{4} = 1$$

so, yes, this is a normalized state. This state is an example of a general superposition over energy eigenstates

$$\Psi(x,t) = \sum_n \psi_n(x)e^{i(n+1/2)\omega t}.$$

The probability of measuring the ground state energy, $\hbar\omega/2$ is the square of the amplitude c_0

$$|c_0|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}.$$

The probability of measuring the $n=1$ state energy, $3\hbar\omega/2$ is

$$|c_1|^2 = \left|\frac{1-i}{2\sqrt{2}}\right|^2 = \frac{1}{4}.$$

The probability of measuring the $n=2$ state energy, $5\hbar\omega/2$ is 0 since it is not part of the superposition. These probabilities do not vary in time. For example one way to show this is to find the time dependent amplitude

$$\langle 1 \mid \Psi\rangle = \frac{1-i}{2\sqrt{2}}e^{i3\omega t/2} = c_1 e^{i3\omega t/2}$$

The probability is the magnitude of this amplitude

$$|\langle 1 \mid \Psi\rangle|^2 = |c_1|^2 = \frac{1}{4},$$

as above. Notice the phase doesn't affect the result.

(c) The expectation value of energy is

$$\langle E\rangle = \sum_n E_n P_n = E_o|c_0|^2 + E_1|c_1|^2 = \hbar\omega\left(\frac{1}{2}\cdot\frac{3}{4} + \frac{3}{2}\cdot\frac{1}{4}\right) = \frac{3}{4}\hbar\omega.$$

The uncertainty in energy is

$$\langle\Delta E\rangle = \langle E^2\rangle - \langle E\rangle^2$$

so we need to compute $\langle E^2\rangle$,

$$\langle E^2\rangle = \sum_n E_n^2 P_n = E_o^2|c_0|^2 + E_1^2|c_1|^2 = (\hbar\omega)^2\left(\frac{1}{4}\cdot\frac{3}{4} + \frac{9}{4}\cdot\frac{1}{4}\right) = \frac{3}{4}(\hbar\omega)^2.$$

Hence,

$$\langle\Delta E\rangle^2 = \frac{3}{4}(\hbar\omega)^2 - \left(\frac{3}{4}\hbar\omega\right)^2 = \frac{3}{16}(\hbar\omega)^2 \implies \langle\Delta E\rangle = \frac{\sqrt{3}}{4}\hbar\omega.$$

(6) Time dependence for a particle in a box.

(a) Comparing the given wavefunction Ψ and the energy eigenstates of the particle in a box $\psi_n(x)$ we see that

$$\Psi(x) = c_1\psi_1(x) + c_2\psi_2(x) \text{ where } c_1 = \frac{1+i}{2} \text{ and } c_1 = \frac{1}{\sqrt{2}}$$

on the interval $0 < x < L$. So the time dependent form has the appropriate phases for each energy:

$$\Psi(x,t) = c_1\psi_1\, e^{-iE_1 t/\hbar} + c_2\psi_2\, e^{-iE_1 t/\hbar} = c_1\psi_1\, e^{-i\pi^2\hbar t/2mL^2} + c_2\psi_2\, e^{-i2\pi^2\hbar t/mL^2}$$

(b) This energy is E_1. From above the probability amplitude is

$$c_1(t) = \frac{1+i}{2}e^{-iE_1 t/\hbar}.$$

At any time the probability is

$$|c_1(t)|^2 = \frac{(1+i)(1-i)}{4} = \frac{1}{2},$$

since the time dependence is just an overall phase.

(c) The expectation value is

$$\langle E\rangle = \sum_n |c_n|^2 E_n = |c_1|^2 E_1 + |c_2|^2 E_2 = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2mL^2} + \frac{4\pi^2\hbar^2}{2mL^2}\right) = \frac{5\pi^2\hbar^2}{4mL^2}.$$

the average of the two energies.

(d) Hmm, $\langle x\rangle$ is

$$\langle x\rangle = \int_0^L \Psi^*(x,t)\, x\, \Psi(x,t)dx = \int_0^L \left(c_1\psi_1\, e^{-iE_1 t/\hbar} + c_2\psi_2\, e^{-iE_1 t/\hbar}\right)^* x \left(c_1\psi_1\, e^{-iE_1 t/\hbar} + c_2\psi_2\, e^{-iE_1 t/\hbar}\right) dx,$$

which has cross terms containing exponentials with $\Delta E = E_1 - E_2$, so it does look time dependent.

I wonder why John didn't ask us to find the dependence? In fact it is a bit of work and something like

$$\langle x\rangle = L\left[\frac{1}{2} - \frac{8\sqrt{2}}{(3\pi)^2}\left(\cos(\Delta E t/\hbar) + \sin(\Delta E t/\hbar)\right)\right]$$

(7) The TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\Psi=E\psi,$$

(a) Let's first divide by $\frac{\hbar^2}{2m}$ which gives

$$-\frac{d^2\psi}{dx^2}+\frac{2mV(x)}{\hbar^2}\psi=\frac{2mE}{\hbar^2}\psi.$$

Letting $x=ya$ gives

$$dx=a\,dy \text{ so that } \frac{d}{dx}=\frac{dy}{dx}\frac{d}{dy}=\frac{1}{a}\frac{d}{dy} \text{ and } \frac{d^2}{dx^2}=\frac{1}{a^2}\frac{d^2}{dx^2}$$

Start from the usual form of the time independent Schrödinger equation for a particle of mass m and energy E in a finite depth potential well of width a. See section 4.1 for the potential. Make the substitution $x=ya$, where y is dimensionless position, and show that the equation above can be rewritten as

$$-\frac{1}{a^2}\frac{d^2\psi}{dy^2}+\frac{2mV(x)}{\hbar^2}\psi=\frac{2mE}{\hbar^2}\psi.$$

Once we multiply through by a^2 and replace $V(x)$ with the depth of the potential well V_o we have the result

$$-\frac{d^2u}{dy^2}+Wu=\epsilon u \tag{1}$$

where

$$W=\frac{2ma^2V_o}{\hbar^2} \text{ and } \epsilon=\frac{2ma^2E}{\hbar^2}.$$

(b) Working in SI units W is

$$[W]=\frac{\text{kg m}^2\text{ J}}{\text{J}^2\text{ s}^2}=\frac{\text{kg m}^2}{\text{J s}^2}=1.$$

So it is dimensionless. The calculation for ϵ is identical.