Quantum (PHYS 290): Guide 7 Solutions
Fall 2025 v1.0

Solutions

(1) Part I: Solving the quantum bouncing ball!

(a) The gravitational potential energy for a ball bouncing off a floor in Earth's gravitational field is "mgh" or

$$V = mgx \text{ for } x > 0 \text{ and } V \to \infty \text{ at } x = 0,$$

with positive x is "up".

(b) The TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\Psi = E\psi,$$

or

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi - \frac{2m^2 g\, x}{\hbar^2}\psi = 0.$$

Letting

$$x = az + b \text{ with } a^3 = \frac{\hbar^2}{2m^2 g} \text{ and } b = \frac{E}{mg}$$

so that $dx = adz$. The TISE becomes

$$\frac{1}{a^2}\frac{d^2\psi}{dz^2} + \left[\frac{2mE}{\hbar^2} - \frac{2m^2 g}{\hbar^2}(az + b)\right]\psi = 0.$$

Substituting in b gives

$$\frac{1}{a^2}\frac{d^2\psi}{dz^2} + \left[\frac{2mE}{\hbar^2} - \frac{2m^2 g}{\hbar^2}(az) - \frac{E}{mg}\frac{2m^2 g}{\hbar^2}\right]\psi = 0.$$

The first and last terms in the square brackets cancel and we have now

$$\frac{1}{a^2}\frac{d^2\psi}{dz^2} + \left[-\frac{az}{a^3}\right]\psi = 0,$$

or

$$\frac{d^2u}{dz^2} - zu = 0 \text{ as desired.} \tag{1}$$

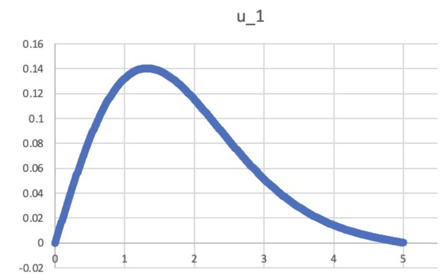
Goal!

(2) (2 pts.) Energies and the wavefunctions for the quantum bouncing ball: I found solutions by using numerical integration to solve the TISE in the form of equation 1 from large z to $z = 0$. At the origin $u = 0$ due to the infinite potential. Using solver I found $E_1 \simeq 2.34$, $E_2 \simeq 4.09$, $E_3 \simeq 5.54$, and $E_2 \simeq 6.79$. These depend on where I started the integration from; if I didn't go far enough out in z the energy eigenvalues were off. The four lowest dimensionless energies are about 2.34, 4.09, 5.52, and 6.79.[1]

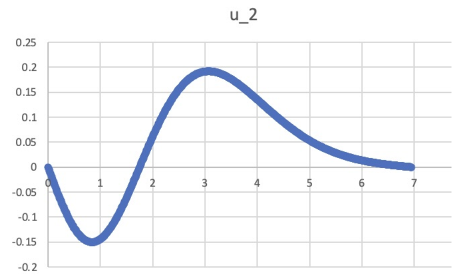
In office hours I saw nice solutions that solved the equation once from positive to negative z. Every crossing at $z < 0$ corresponds to a solution. This method worked very well.

The unnormalized wavefunctions are shown in the figure on the next page. Plot (a) is the ground state, (b) is the first excited state, etc. Notice that I had to go to higher z as the wavefunctions stretched out. I didn't go to high enough z for u_3 leading to the difference above between the computed and exact solutions.

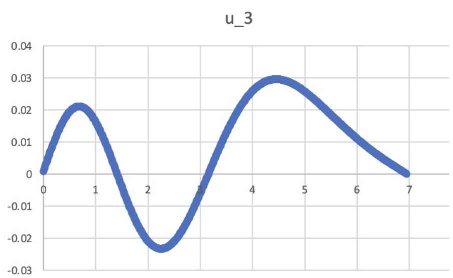
[1] The exact values are obtained from the zeros of the Airy function, the dimensionless energies b/a when $Ai(-b/a) = 0$.



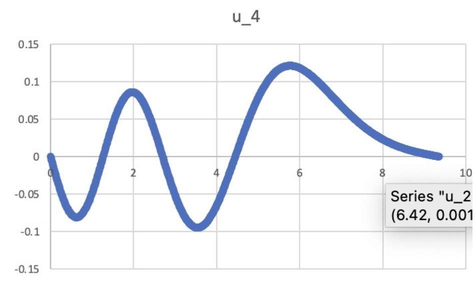
(a.)



(b.)



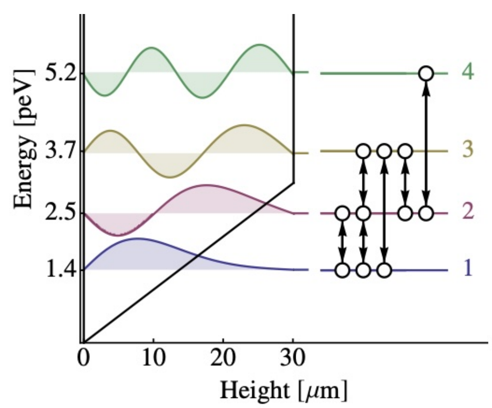
(c.)



(d.)

Graders: For full credit solutions must include the correct energies and plots, not sketches.

(3) To explain some of



we should address wavefunctions and energies.

- The wavefunctions are just what we found in the previous problem - well, up to the $n = 3$ and $n = 4$ levels, where it looks like they switched to square well potential.
- At the point where the potential switches, the lower part of the well has a width of 30 μm and a height of approximately 3.0 peV. Is this mgh? Yes, since

$$m_n gh = 940 \text{ MeV}/c^2 \cdot 9.8 \text{ m/s}^2 \cdot 30\mu\text{m} \;= 3.0 \text{ peV, as in the figure.}$$

  This gives me hope that the eigenvalues for the $n = 1$ and $n = 2$ states should be good matches.

- In the last problem we found the dimensionless energies b/a so these should correspond to the energies in the figure. For the ground state
$$2.34 \simeq \left.\frac{b}{a}\right|_1 = \left(\frac{E_1}{m_n g}\right)\left(\frac{2m_n^2 g}{\hbar^2}\right)^{1/3} \implies E_1 \simeq 2.34\left(\frac{\hbar^2 m_n g^2}{2}\right)^{1/3}$$
which should be 1.4 peV but as I type this up I find 4.8 peV, way off. For the first excited state
$$4.09 \simeq \left.\frac{b}{a}\right|_2 = \left(\frac{E_2}{m_n g}\right)\left(\frac{2m_n^2 g}{\hbar^2}\right)^{1/3} \implies E_2 \simeq 4.09\left(\frac{\hbar^2 m_n g^2}{2}\right)^{1/3}$$
which should be 2.5 peV.

For full credit this one needs the discussion of the wavefunctions and the two energies levels. Correct solutions do not need the comments on the potential or transitions between levels

(4) On parity

(a) To show that the parity operator is hermitian let's apply the definition for any operator $\hat{A}$,
$$\int_{-\infty}^{\infty} \phi(x)^* \left(\hat{A}\psi(x)\right) dx = \int_{-\infty}^{\infty} \left(\hat{A}\phi(x)\right)^* \psi(x) dx.$$
Denoting the parity operator $\hat{\Pi}$ the left hand side is
$$\int_{-\infty}^{\infty} \phi(x)^* \hat{\Pi}\psi(x) dx = \int_{-\infty}^{\infty} \phi(x)^* \psi(-x) dx.$$
The right hand side is
$$\int_{-\infty}^{\infty} \left(\hat{\Pi}\phi(x)\right)^* \psi(x) dx = \int_{-\infty}^{\infty} \left(\phi(-x)\right)^* \psi(x) dx.$$
Letting $y = -x$, so that the integral limits swap and $dy = -dx$, the right hand side becomes
$$\int_{-\infty}^{\infty} \left(\phi(-x)\right)^* \psi(x) dx = -\int_{\infty}^{-\infty} \phi(y)^* \psi(-y) dy = \int_{-\infty}^{\infty} \phi(y)^* \psi(-y) dy$$
which is equal to the left hand side. Thus, $\hat{\Pi}$ is hermitian.

(b) Letting $i, j = \pm$ and p_i be the eigenvalues then from above
$$\int_{-\infty}^{\infty} \psi_i(x)^* \hat{\Pi}\psi_j(x) dx = p_j \int_{-\infty}^{\infty} \psi_i(x)^* \psi_j(x) dx$$
and
$$\int_{-\infty}^{\infty} \psi_i(x)^* \hat{\Pi}\psi_j(x) dx = p_i \int_{-\infty}^{\infty} \psi_i(x)^* \psi_j(x) dx.$$
Subtracting gives
$$(p_i - p_j) \int_{-\infty}^{\infty} \psi_i(x)^* \psi_j(x) dx = 0$$
so the inner product vanishes when the eigenvalues are different; they are orthogonal.

This can be done in Dirac notation. So for cultural interest ...the eigenvalues of the parity operator are ± 1. So
$$\hat{\Pi} \mid +\rangle = + \mid +\rangle \text{ and } \hat{\Pi} \mid -\rangle = - \mid -\rangle.$$

Letting $i, j = \pm$ and p_i be the eigenvalues then these two expressions may be written as

$$\hat{\Pi} \mid i\rangle = p_i \mid i\rangle.$$

Now since $\langle i \mid \hat{P}i \mid j\rangle$ is equal to both

$$\langle i \mid \left(\hat{\Pi} \mid j\rangle\right) = p_j \langle i \mid j\rangle$$

and

$$\langle i \mid \hat{\Pi} \mid j\rangle = \langle i \mid \hat{\Pi}^\dagger \mid j\rangle = p_i \langle i \mid j\rangle,$$

we have

$$p_j \langle i \mid j\rangle = p_i \langle i \mid j\rangle \implies (p_i - p_j)\langle i \mid j\rangle = 0$$

so either we have the same state and eigenvalue in which case $i = j$ and $\langle i \mid j\rangle = 1$ (for normalized states) or we have different states $i \neq j$ and $\langle i \mid j\rangle = 0$. Thus, $\langle i \mid j\rangle = \delta_{ij}$.

(5) On derivatives

(a) To show that the derivative operator is not hermitian let's apply the definition for any operator $\hat{A}$,

$$\int_{-\infty}^{\infty} \phi(x)^* \left(\hat{A}\psi(x)\right) dx = \int_{-\infty}^{\infty} \left(\hat{A}\phi(x)\right)^* \psi(x) dx.$$

On integration by parts the derivative acting on ψ is

$$\int_{-\infty}^{\infty} \phi(x)^* \frac{d}{dx}\psi(x) dx = [\phi^*\psi]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\phi(x)^*}{dx}\psi(x) dx.$$

If we assume the particle is somewhat localized and so wavefunctions vanish at $\pm\infty$ then we have

$$\int_{-\infty}^{\infty} \phi(x)^* \frac{d}{dx}\psi(x) dx = -\int_{-\infty}^{\infty} \frac{d\phi(x)^*}{dx}\psi(x) dx.$$

which is *not* equal to

$$\int_{-\infty}^{\infty} \frac{d\phi(x)^*}{dx}\psi(x) dx$$

so so the derivative fails to be hermitian.

(b) BUT! if the operator is instead $-id/dx$ then acting to the right and integrating by parts

$$\int_{-\infty}^{\infty} \phi(x)^* \left(-i\frac{d}{dx}\psi(x)\right) dx = +i\int_{-\infty}^{\infty} \frac{d\phi(x)^*}{dx}\psi(x) dx.$$

which *is* equal to

$$\int_{-\infty}^{\infty} \left(-i\frac{d}{dx}\phi(x)\right)^* \psi(x) dx = +i\int_{-\infty}^{\infty} \frac{d\phi(x)^*}{dx}\psi(x) dx$$

and so this operator is hermitian. Notice $\hbar$ plays no role on this - it just selects which units were using.