Quantum (PHYS 290): Guide 8 Solutions

Fall 2025 v1.0

Solutions:

(1) Let's operate with $\hat{p}$ to see which functions are eigenfunctions. Starting with u_1

$$\hat{p}u_1 = -i\hbar\frac{du_1}{dx} = -i\hbar k\cos(kx) \neq pu_1,$$

so u_1 is not an eigenfunction of momentum. For u_2,

$$\hat{p}u_2 = -i\hbar\frac{du_2}{dx} = (-i)^2\hbar k e^{-ik(x-a)} = p_2 u_2,$$

so u_2 is an eigenfunction of $\hat{p}$ with eigenvalue $p_2 = -\hbar k$.

$$\hat{p}u_3 = -i\hbar\frac{du_3}{dx} = -i\hbar k\cos(kx) + i\hbar k\sin(kx) \neq pu_3,$$

so u_3 is not an eigenfunction of momentum. Finally,

$$\hat{p}u_4 = -i\hbar\frac{du_4}{dx} = i\hbar k\sin(kx) + \hbar k\cos(kx) = \hbar k\left[\cos(kx) + i\sin(kx)\right] = p_4 u_4,$$

where $p_4 = \hbar k$.

(2) A quick integral version:

$$\left\langle A^2 \right\rangle = \int \psi^* \hat{A}^2 \psi dx = \int \left(\hat{A}\psi\right)^* \left(\hat{A}\psi\right) dx$$

The last equality uses the fact that $\hat{A}$ is Hermitian. Defining a new wavefunction $\varphi = \hat{A}\psi$ the expectation value becomes

$$\left\langle A^2 \right\rangle = \int \left(\hat{A}\psi\right)^* \left(\hat{A}\psi\right) dx = \int \varphi^* \varphi dx \geq 0$$

as desired.

For fun, a longer Dirac notation version: If $\hat{A}$ is hermitian then it has real eigenvalues a_n and

$$\hat{A} \mid n\rangle = a_n \mid n\rangle.$$

The states $\mid n\rangle$ are the eigenfunctions. Since these states are complete, we can use

$$1 = \sum_n \mid n\rangle\langle n \mid .$$

For *any* state

$$\mid \psi\rangle = \sum_n c_n \mid n\rangle,$$

the expectation value $\left\langle \hat{A}^2 \right\rangle$ is

$$\begin{aligned}
\left\langle \hat{A}^2 \right\rangle &= \langle \psi \mid \hat{A}\cdot\hat{A} \mid \psi\rangle = \sum_n \langle \psi \mid \hat{A} \mid n\rangle\langle n \mid \hat{A} \mid \psi\rangle \\
&= \sum_n a_n^2 \langle \psi \mid n\rangle\langle n \mid \psi\rangle \\
&= \sum_n a_n^2 |c_n|^2 \geq 0
\end{aligned}$$

since a^2 and $|c_n|^2$ are clearly non-negative.

(3) This version requires the property $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$. If the commutator of two Hermitian operators $\hat{A}$ and $\hat{B}$,

$$\left[\hat{A},\hat{B}\right] = i\hat{C}$$

- note the i! - then

$$\left[\hat{A},\hat{B}\right]^\dagger = \left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)^\dagger = \left(\hat{A}\hat{B}\right)^\dagger - \left(\hat{B}\hat{A}\right)^\dagger$$

but from the above property

$$\left(\hat{A}\hat{B}\right)^\dagger - \left(\hat{B}\hat{A}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger = \hat{B}\hat{A}-\hat{A}\hat{B}.$$

The last equality follows since both operators are Hermitian. Hence,

$$\left[\hat{A},\hat{B}\right]^\dagger = -\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right) = -\left[\hat{A},\hat{B}\right]$$

So

$$\left[\hat{A},\hat{B}\right]^\dagger = -\left[\hat{A},\hat{B}\right] \text{ and } \left(i\hat{C}\right)^\dagger = -i\hat{C} = i\hat{C}$$

Hence,

$$\hat{C} = \hat{C}^\dagger.$$

(4) In class we had the difference (or deviation)

$$\hat{D}_A = \hat{A} - \langle A\rangle\,.$$

(a) The square of the expectation value is

$$\begin{aligned}\left\langle \hat{D}_A^2\right\rangle &= \left\langle\left(\hat{A}-\langle A\rangle\right)^2\right\rangle\\ &= \left\langle\left(\hat{A}^2 - 2\hat{A}\langle A\rangle + \langle A\rangle^2\right)\right\rangle\\ &= \left\langle\hat{A}^2\right\rangle - 2\left\langle\hat{A}\right\rangle\langle A\rangle + \langle A\rangle^2 \text{ since } \left\langle\hat{A}\right\rangle = \langle A\rangle\\ &= \left\langle A^2\right\rangle - \langle A\rangle^2 = \left\langle\Delta A^2\right\rangle = \Delta A^2\end{aligned}$$

so the expectation value of the square of the deviation of A is the uncertainty in A squared.

(b) First, since scalar multiplication is commutative $[a,b]=0$ for any two numbers a and b. Also for the same reason if we have a number a and an operator $\hat{B}$ then

$$\left[a,\hat{B}\right] = a\hat{B} - \hat{B}a = a\hat{B} - a\hat{B} = 0$$

Now rom the definition

$$\begin{aligned}\left[\hat{D}_A,\hat{D}_B\right] &= \left[\hat{A}-\langle A\rangle\,,\hat{B}-\langle B\rangle\right]\\ &= \left[\hat{A},\hat{B}\right] - \left[\hat{A},\langle B\rangle\right] - \left[\langle A\rangle\,,\hat{B}\right] + \left[\langle A\rangle\,,\langle B\rangle\right] \text{ from above}\\ &= \left[\hat{A},\hat{B}\right]\end{aligned}$$

as expected.

(5) Tunneling for electrons leaving a metal

(a) From the diagram in (b) the potential equals $E_f + W$ at $x = 0$ then it falls linearly with x so

$$V(x) = E_f + W - e|\mathbf{E}|x$$

where $\mathbf{E}$ is the electric field and e is the fundamental charge.

(b) Using the potential above and energy E_f the approximate tunneling transmission is

$$T \simeq \exp\left(-\frac{2}{\hbar}\int\sqrt{2m(E_f + W - e|\mathbf{E}|x - E_f)}dx\right) = \exp\left(-\frac{2}{\hbar}\int\sqrt{2m(W - e|\mathbf{E}|x)}dx\right)$$

Now we need the limits of integration. The barrier extends from $x = 0$ to x_* when the potential equals the electron energy:

$$E_f + W - e|\mathbf{E}|x_* = E_f \implies x_* = \frac{W}{e|\mathbf{E}|}$$

So the integral is

$$\sqrt{2m}\int_0^{W/e|\mathbf{E}|}\sqrt{W - e|\mathbf{E}|x}\,dx.$$

Changing variables to $y = W - e|\mathbf{E}|x$ so that

$$dx = -\frac{dy}{e|\mathbf{E}|} \text{ and}$$

and $y = W$ when $x = 0$ and $y = 0$ when $x = x_*$. The integral becomes

$$-\frac{\sqrt{2m}}{e|\mathbf{E}|}\int_W^0\sqrt{y}\,dy = \frac{\sqrt{2m}}{e|\mathbf{E}|}\int_0^W\sqrt{y}\,dy = \frac{\sqrt{2m}}{e|\mathbf{E}|}\frac{2}{3}W^{3/2}.$$

So the transmission is

$$T \simeq \exp\left(-\frac{2}{\hbar}\frac{\sqrt{2m}}{e|\mathbf{E}|}\frac{2}{3}W^{3/2}\right) = \exp\left(-\frac{4\sqrt{2mW^3}}{3e|\mathbf{E}|\hbar}\right)$$

as expected.

(6) Hydrogen energy levels

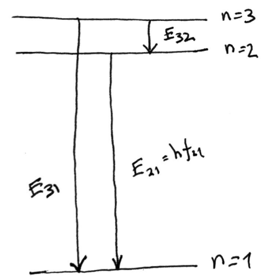
(a) From $E_N = -13.6\text{ eV}/n^2$ the energies are about $E_1 = -13.6, E_2 = -3.4$, and $E_3 = -1.5$ all in electron volts.

(b) For outgoing photons the transitions are 2 −>1, 3 −>1, and 3 −>2. So in this order

$$E_{21} \simeq 10.2\text{ eV} = hf \implies f_{21} \simeq 2.47 \times 10^{15}\text{ Hz, and } \lambda_{21} \simeq \frac{c}{f_{21}} \simeq 122\text{ nm}$$
$$E_{31} \simeq 12.1\text{ eV} = hf \implies f_{31} \simeq 2.92 \times 10^{15}\text{ Hz, and } \lambda_{31} \simeq \frac{c}{f_{31}} \simeq 102\text{ nm}$$
$$E_{32} \simeq 1.9\text{ eV} = hf \implies f_{32} \simeq 4.57 \times 10^{14}\text{ Hz, and } \lambda_{32} \simeq \frac{c}{f_{32}} \simeq 656\text{ nm}$$

Calculate the frequencies in Hz and the wavelengths in nm of all the photons that can be emitted from the atom in transitions between these levels.

(c) A sketch -



(d) Looking up the wavelengths on the plot given in the guide, 2 −>1, 3 −>1 are in the ultra-violet while 3 −>2 is in the visible. It's red.