Quantum (PHYS 290): Guide 9

Fall 2025 v1.0

Intro:

In our choose-your-own adventure in angular momentum, we delved into the details of the angular solution to the TISE in 3D. The approach is through ladder operators rather than solutions to differential equations. By the end of the class on Friday November 14 we'll have a complete solution to the question of hydrogen atom wavefunctions. (We numerically solved for the low lying radial solutions in lab.)

Reading: ("T" stands for Townsend's text)

- So far we have discussed Townsend's Sections 6.2 and 6.3. We will skip (at least for now) section 6.4
- The ladder operators are discussed in section 6.2 of Schoeder's quantum book.
- Looking ahead - Chapter 6 Section 5, Chapter 5 Section 6.

Extensions this week are due Monday morning at 10 AM. Please drop them by or email them.

Problems: Due Friday, November 21 at the beginning of class

The first set of problems supliment our discussion of the angular.

(1) From the position-momentum uncertainty relations and the definition of the angular momentum operators show that

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar \hat{L}_x.$$

So we now have the relations

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z,\ \left[\hat{L}_y, \hat{L}_z\right] = i\hbar \hat{L}_x,\ \text{and}\ \left[\hat{L}_z, \hat{L}_x\right] = i\hbar \hat{L}_y$$

(2) Show that $[\hat{L}^2, \hat{L}_z] = 0$. Explain why $[\hat{L}^2, \hat{L}_y] = 0$ and $[\hat{L}^2, \hat{L}_x] = 0$ as well. Hint: Use $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and $[AB, C] = A[B, C] + [A, C]B$ for operators A, B, C.

(3) Derive the commutator relations

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hbar \hat{L}_\pm.$$

(4) Show that if $\mid \psi\rangle$ is an eigenstate of then

$$\hat{L}_z \left(\hat{L}_- \mid \psi\rangle\right) = \hbar(m-1)\left(\hat{L}_- \mid \psi\rangle\right).$$

State briefly what this relation means.

(5) (Optional worth 1 extra pt.) 6.8 The form of $\hat{L}_x$ and $\hat{L}_y$ in spherical coordinates.

(6) By using the definition of the ladder operators in terms of $\hat{L}_x$ and $\hat{L}_y$, show that the ladder operators have a angular space representation of

$$\hat{L}_{\pm} = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

If you did not complete the last problem the form of the operators are given in equations (6.27) and (6.28) in Townsend. Please do the "up" operator $\hat{L}_+$ first then discuss the "down" one.

(7) Apply the lowering operator to obtain a general formula for $Y_{\ell\,\ell-1}(\theta,\varphi)$. Apply this and the ladder operator to find the three states for $\ell = 2$. Check your results equations using (6.47) - (6.49) in Townsend. Don't worry about normalization constants.

(8) 6.15 Applying our angular solutions to a new setting.

(9) 6.26 Practice with matrix multiplication - and the Pauli matrices